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On Fairness in Social Decision Schemes

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I hereby confirm that this is my own work, and that I used only the cited sources and materials.

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Abstract

Settling on an action of a society by vote of its members is one of the oldest procedures of civilization, even the Athenian democracy and the election of Venice's Doge for example relied upon it. Many ideas on how to determine the course of action based on a society's votes have been developed ever since. While all these decision schemes have been thoroughly analysed in terms of efficiency and manipulability, there exists little to no work on analysing fair treatment of each voter apart from very simple requirements. We will study the difficulties of this topic and propose a new fairness condition for voting rules. We also will apply this definition to known voting systems and introduce new algorithms which incorporate this view on fairness.

Zusammenfassung

Eine gemeinschaftliche Entscheidung basierend auf Wahlen zu treffen ist eine der ältesten Vorgehensweisen der Zivilisation. Beispielsweise wurden bereits in der athenischen Demokratie oder der Wahl des Doge von Venedig komplexe Wahlsysteme eingesetzt. Seit jeher wurden viele verschiedene Wege den Ausgang einer Wahl zu bestimmen erdacht. Diese Systeme wurden sehr gründlich in Bezug auf ihre Effizienz und Manipulierbarkeit analysiert, jedoch existiert kaum Forschung bezüglich fairer Behandlung der Wählenden untereinander. In dieser Arbeit gehe ich auf die Schwierigkeiten dieses Themas ein und definiere eine neue Fairness-Bedingung für Wahlsysteme. Außerdem werde ich diese Definition auf bereits bekannte Wahlsysteme anwenden und neue Algorithmen einführen, die mit dieser Definition im Sinn entwickelt wurden.

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Chapter 1

Introduction

In this chapter, we briefly explain the basics of social choice theory and give a short overview of the current state of research.

1.1 Preliminaries

Broadly speaking, the motivation of social choice theory is to explain how a society of multiple acting entities should settle on a common decision based on the preferences of each entity. This process can loosely be described as "voting". The term might be a bit misleading, as many results of this theory are not only used in classical voting, but in other fields, too, for example in computer science when dealing with multi-agent systems or swarm intelligence (we recommend reading [BCE12] for an in depth introduction to this topic). In the following, we will explain how a voting process is usually captured in mathematical terms, introduce basic properties and provide some examples of known voting procedures.

The Society

To have votes, there obviously have to be some voters first. Let thus $n \in \mathbb{N}$ be the number of voters, $N = \{1, \dots, n\}$ the society (i.e. the set of all entities participating in the vote), also called *agents*. The next elementary component of voting is a set of alternatives between which the agents have to choose, denoted by A, which also is assumed to be finite. One may differentiate between the options ("strategies") an agent is able to pick from and the consequences ("outcomes") such an option has, but these two concepts are almost always identified and not discussed further here.

Agents are assumed to have complete and transitive preferences over these alternatives (an agent might very well be completely indifferent between all of them). Seemingly trivial at first glance, this requirement is not to be taken for granted at all - human preferences often are neither transitive nor complete (cf. [Fis82b; Fis91]). However, exploring this topic further is beyond the scope of this work. Every agent's preferences then are represented by complete and transitive relations $R \subset A \times A$. The modelling is understood as follows: If $(a, b) \in R$, denoted by $a\ R\ b$, the agent values alternative a

the same as or higher than b. While there is a subtle difference between the two terms, we identify the preferences of an agent with the corresponding preference relation.

Conforming to usual notation, for given a preference relation R, P describes the strict part of R ($a\ P\ b$ iff $a\ R\ b$ but not $b\ R\ a$) and I the indifference under R ($a\ I\ b$ iff $a\ R\ b$ and $b\ R\ a$). The entirety of every agents' preferences is represented by $\mathcal{R} = (R_1, \ldots, R_n)$, $\mathcal{R}$ is called *preference profile* (or just *profile*). For $i \in N$ and a profile $\mathcal{R}$, R_{-i} loosely denotes the tuple of all preferences except agent i's. The notation of *equivalence classes* allows for a readable representation of preferences:

Definition 1.1 (Maximal elements & equivalence classes)
Let R be any preference relation over the objects A. The maximum elements of a set $B \subset A$ relative to a preference relation R is defined as

$$\max_R B := \{a \in B : a\ R\ b\, \forall\, b \in B\}.$$

The most preferred equivalence class $E(R, 1)$ is the set of all "best" alternatives, i.e. $E(R, 1) := \max_R A$. For $k \geq 2$, the k-th equivalence class is defined as the set of the best alternatives among those which are not in any previous equivalence class: $E(R, k) := \max_R(A \setminus \bigcup_{j=1}^{k-1} E(R, k))$. Note that by completeness $a\ I\ b$ iff $\exists\, k : a, b \in E(R, k)$. Additionally, $m(R) := \max\{k \in \mathbb{N} : E(R, k) \neq \emptyset\}$ is the number of equivalence classes induced by R. If it is clear from the context which preference R is talked about, $E(R, k)$ and $m(R)$ may be abbreviated by E^k and m.

We will often use the term *primary preferences* or *primary preference class* to refer to $E(R, 1)$ of some preference relation R. As equivalence classes fully characterize a preference relation R, preferences can be represented in a readable manner by its equivalence classes: $R : E^1, E^2, \ldots, E^m$. For example (with $A = \{a, b, c\}$), the preference specified by $c\ P\ b$, $b\ I\ a$ may be written as $\{c\}, \{a, b\}$.

To cast their vote, every agent will (simultaneously) submit a preference relation to some decision device. It is important to notice that agents are not bound to submit their true preferences but indeed may report any (allowed) preference. The complete information about the society and their preferences on a certain matter are contained in a *vote* $\mathfrak{V} := (N, A, \mathcal{R})$. Listing each agent's equivalence classes according to the previously defined notation suffices to represent a vote:

$$\begin{aligned} 1 &: E_1^1, \ldots, E_1^{m_1} \\ &\vdots \\ n &: E_n^1, \ldots, E_n^{m_n} \end{aligned}$$

where E_i^k and m_i is shorthand for $E(R_i, k)$ and $m(R_i)$.

Decision Schemes & Lotteries

Now that every agent has submitted his vote, these votes have to be processed in a (beforehand agreed on) manner to reach a decision of the whole society. This can be done in several ways, one approach used nowadays are randomized social choice functions, also called *social decision schemes* (SDS) ([Gib77; Bar79]). These are functions which, given a vote, output a *lottery* over all alternatives. If in the end a single action is to be found, an appropriate device of randomness is used to select an alternative according to the returned distribution.

Definition 1.2 (Lotteries)
The set of all lotteries (probability distributions) over a set A is denoted by $\triangle(A)$,

$$\triangle(A) := \left\{ p : A \to [0,1], \sum_{a \in A} p(a) = 1 \right\}.$$

The *support* of a lottery p, written as $\widehat{p}$, is defined as the set of all alternatives receiving a non-zero probability, $\widehat{p} := \{a \in A : p(a) > 0\}$. For $k \in \mathbb{N}$, $\lambda \subset [0,1]^k$ with $\sum_{j=1}^{k} \lambda_j = 1$ and $p \subset \triangle(A)^k$, the λ *convex combination* of the lotteries p_j is defined as

$$\left(\sum_{j=1}^{k} \lambda_j p_j \right)(a) := \sum_{j=1}^{k} \lambda_j p_j(a).$$

Lotteries with $|\widehat{p}| = 1$ are called *degenerate* and are identified with the corresponding alternative. This way, every lottery p can be written as a convex combination of $|\widehat{p}|$ degenerate lotteries: $p = \sum_{a \in \widehat{p}} p(a) \cdot a$.

Definition 1.3 (Social decision scheme & correspondence)
A social decision scheme (SDS) is a function that maps a vote $\mathfrak{V} = (N, A, \mathcal{R})$ to a lottery $p \in \triangle(A)$, i.e. $f(\mathfrak{V}) \in \triangle(A)$. If f instead yields a set of lotteries, it is called *correspondence* ($f(\mathfrak{V}) \subseteq \triangle(A)$). A correspondence g is called *sub-correspondence* of f, if $g(\mathfrak{V}) \subseteq f(\mathfrak{V})$ for every $\mathfrak{V}$.

We identify SDS with correspondences yielding a single-element set. The need for correspondences is rather technical, but required and will become apparent throughout the thesis. Most of the correspondences dealt with here are *essentially single-valued*, meaning that every agent is indifferent between any of the returned lotteries.

Definition 1.4 (Essentially single-valued correspondence)
A correspondence f is called essentially single-valued, if for every $\mathfrak{V} = (N, A, \mathcal{R})$, agent $i \in N$, $k \in \mathbb{N}$ and $p, q \in f(\mathfrak{V})$ one has that $p(E_i^k) = q(E_i^k)$.

One might argue that this inherent randomness contradicts some basic ideas of voting ("What good is a vote if the decision depends on randomness anyway in the end?"), but this probabilistic approach does have several virtues over requiring the scheme to return a single alternative. In the latter case one can easily construct examples which are inherently unfair: Say there are two voters deciding between two alternatives. Should one vote for the first option and the other vote for the second one, using a coin toss to decide between the two alternatives seems to be the only fair way to solve this problem. However, as no probability is allowed, the decision scheme has to intrinsically prefer one alternative over the other.

Gibbard [Gib73] and Satterthwaite [Sat75] independently have shown that every (proper) deterministic decision scheme suffers from this affliction, as the method for breaking ties always has to have some inherent flaws. A further benefit of lotteries is that they can be interpreted as shares of divisible goods. These range from budget spendings over time or space allocations to seats in a parliament. This idea of randomization (surprisingly) is nearly as old as voting itself and nowadays is prevalent in many fields, especially computer science ([BCE12; Pro10; CS06; WX12]) and political research ([Sto11]).

Anonymity and Neutrality

Now that voting procedures are defined, we will introduce some known properties they can fulfil or violate. The most basic conditions are *anonymity* and *neutrality*.

Definition 1.5 (Anonymity and neutrality)
Let $\mathfrak{V} = (N, A, \mathcal{R})$ be a vote and $\pi : N \to N$, $\sigma : A \to A$ permutations of the agents and alternatives respectively. Define $\pi(\mathcal{R}) := (R_{\pi(1)}, ..., R_{\pi(n)})$, i.e. renaming all agents according to π, and $\sigma(\mathcal{R})$ as the preference profile where all alternatives are renamed according to σ. Similarly, for $p \in \triangle(A)$ define $\sigma(p)$ as the lottery obtained from analogous renaming.

A correspondence f is called anonymous, if for every $\mathcal{R}$ and agent-permutation π one has that $f(\mathcal{R}) = f(\pi(\mathcal{R}))$. In other words, it is required that f does not prefer one agent over an other, f treats all agents equally. f is called neutral, if for every $p \in f(\sigma(\mathcal{R}))$ there exists a $q \in f(\mathcal{R})$ with $p = \sigma(q)$. This means that f does not distinguish ex ante between the alternatives.

If not otherwise noted, all SDS/correspondences dealt with are assumed to fulfil these two properties.

Lottery Extensions

Agents have preferences over the objects in A, but correspondences return lotteries. So, to be able to reason about the results of a correspondence, the preference relation of an agent has to be extended to the domain of lotteries in a sensible way. A *lottery extension* χ extends a given preference relation R over A to a (possibly incomplete, but transitive) relation over all lotteries $\triangle(A)$, denoted by $p\ R^\chi\ q$. Today many reasonable extensions are known and used, see [ABB14] for an extensive list and [Cho12] for an in-depth discussion of lottery extensions in general. In this work we will use the following:

Definition 1.6 (Lottery extensions)

- *Sure thing* (ST): Introduced by Aziz, Brandt, and Brill [ABB13a][1], p is said to be ST-better or equal than q, if every alternative in the support of p is either preferred to all in the support of q or it receives the same probability under both lotteries. Formally $p\ R^{\text{ST}}\ q$ if

$$\forall\, a \in \widehat{p} : (a\ P\ b \,\forall\, b \in \widehat{q}) \text{ or } (p(a) = q(a) \text{ and } a\ P\ b \,\forall\, b \in \widehat{q} \setminus \widehat{p})\,.$$

  The underlying idea is to only consider the alternatives for which p and q differ, strongly related to the independence axiom formulated by Neumann and Morgenstern [NM47].

- *Stochastic dominance* (SD): A lottery p is SD-better or equal than q, if for every $a \in A$ the distribution p yields at least as much probability of selecting an alternative equally good or better than a as q. In mathematical terms $p\ R^{\text{SD}}\ q$ if

$$\forall\, a \in A : p(\{b \in A : b\ R\ a\}) \geq q(\{b \in A : b\ R\ a\}).$$

- *Downward lexicographic* (DL): A lottery p is DL-better or equal than q, if either all equivalence classes have an equal probability of being selected or the first equivalence class for which the probabilities do differ between p and q has a higher chance of being selected under p than under q. Formally, $p\ R^{\text{DL}}\ q$ if either $p(E^k) = q(E^k)\,\forall\, k \leq m$ or $p(E^k) > q(E^k)$ for the smallest k with $p(E^k) \neq q(E^k)$. It is worth noting that DL is complete.

Next, we will prove some properties of these lottery extensions. At first a characterization of ST is provided, as the given definition is somewhat unintuitive.

[1] Note that their original definition unfortunately contained an error which will be corrected in the upcoming journal version of this paper.

Proposition 1.7
Let R be any preference relation and $p, q \in \triangle(A)$. Then $p\ R^{\mathrm{ST}}\ q$ iff the following conditions hold:

1. *Every element which exclusively is in the support of p is better than each element in the intersection of the supports and these are better than the ones exclusive to q's support (For each $a \in \widehat{p} \setminus \widehat{q}$, $b \in \widehat{p} \cap \widehat{q}$, $c \in \widehat{q} \setminus \widehat{p}$ it holds that $a\ P\ b$ and $b\ P\ c$).*

2. *Every element in the intersection of the supports has equal probability under p and q ($p(a) = q(a)$ for all $a \in \widehat{p} \cap \widehat{q}$).*

Also, $p\ P^{\mathrm{ST}}\ q$ iff additionally $p \neq q$.

Proof. $\Rightarrow$: Let $p, q \in \triangle(A)$ with $p\ R^{\mathrm{ST}}\ q$. As the proof is simple in the case of $p = q$, assume w.l.o.g. $p \neq q$. Define $A_p := \widehat{p} \setminus \widehat{q}$, $A_\cap := \widehat{p} \cap \widehat{q}$ and $A_q := \widehat{q} \setminus \widehat{p}$. Let $b \in A_q$. By assumption, for all $a \in \widehat{p}$ in both cases it holds that $a\ P\ b$. Let now instead $a \in A_p$. As then clearly $p(a) > 0 = q(a)$, by definition of ST $a\ P\ b \,\forall\, b \in \widehat{q}$. This proves Item 1. To prove Item 2, let $a \in A_\cap$ arbitrary. As clearly not $a\ P\ a$ (note that $a \in \widehat{q}$), the definition of ST requires that $p(a) = q(a)$.

$\Leftarrow$: Let p and q fulfil the two conditions. Choose $a \in \widehat{p}$ arbitrary. If $a \in A_p$, then by Item 1 $a\ P\ b \,\forall\, b \in \widehat{q}$. If on the other hand $a \in A_\cap$, Item 2 gives that $p(a) = q(a)$ and again using Item 1 one has $a\ P\ b \,\forall\, b \in A_q = \widehat{q} \setminus \widehat{p}$.

Now, the second statement is left to prove. In conjunction with the already proven statement it is equivalent to showing that for any two lotteries $p, q \in \triangle(A)$ one has that $p\ R^{\mathrm{ST}}\ q$ and $q\ R^{\mathrm{ST}}\ p$ iff $p = q$: The backward direction is trivial. To prove the forward direction, let p and q be as specified. The already proven Item 1 shows that $A_p = A_q = \emptyset$, as everything else immediately results in a contradiction. Further applying Item 2 yields $p = q$. □

This yields an interesting condition for ST-maximal lotteries:

Corollary 1.8
Let R be an arbitrary preference relation and p a lottery with $p(E^1) > 0$ ($\widehat{p} \cap E^1 \neq \emptyset$). Then there exists no lottery q with $q\ P^{\mathrm{ST}}\ p$.

Proof. Assume that some q with $q\ P^{\mathrm{ST}}\ p$ exists. By Proposition 1.7 we have that $p \neq q$ and $p(a) = q(a)$ for all $a \in \widehat{p} \cap \widehat{q}$. Hence, both $\widehat{p} \setminus \widehat{q}$ and $\widehat{q} \setminus \widehat{p}$ are not empty: Suppose that only $\widehat{q} \setminus \widehat{p}$ is not empty, hence $q(\widehat{p}) < 1$. However, as $\widehat{p} \subset \widehat{q}$, $p(\widehat{p}) = q(\widehat{q}) < 1$, which is a contradiction. The same argument can be employed for $\widehat{p} \setminus \widehat{q}$.

Proposition 1.7 additionally yields that every element in $\widehat{q} \setminus \widehat{p}$ is strictly preferred over any element of $\widehat{p} \cap \widehat{q}$. Especially, there is no "best element" in the intersection of the supports ($\widehat{p} \cap \widehat{q} \cap E^1 = \emptyset$). On the other hand, the assumption $p(E^1) > 0$ implies $\widehat{p} \cap E^1 \neq \emptyset$. Together this requires that $(\widehat{p} \setminus \widehat{q}) \cap E^1 \neq \emptyset$. In other words, there is a best element in $\widehat{p} \setminus \widehat{q}$, which is a contradiction to Proposition 1.7. □

Now, we provide an intuitive characterization of SD.

Corollary 1.9
Let R be a preference relation and $p, q \in \triangle(A)$ lotteries. Then $p\ R^{\mathrm{SD}}\ q$ iff

$$p\left(\bigcup_{i=1}^{j} E^i\right) = \sum_{i=1}^{j} p\left(E^i\right) \geq \sum_{i=1}^{j} q\left(E^i\right) = q\left(\bigcup_{i=1}^{j} E^i\right) \quad \forall\, j \leq m.$$

Proof. Let $a \in A$ be arbitrary, choose j such that $a \in E^j$. One has that

$$\{b \in A : b\ P\ a\} = \{b \in A : b\ I\ a\} \cup \{b \in A : b\ P\ a\} = E^j \cup \bigcup_{i=1}^{j-1} E^i = \bigcup_{i=1}^{j} E^i$$

and $a, a' \in E^j \Leftrightarrow \{b \in A : b\ R\ a\} = \{b \in A : b\ R\ a'\}$. The statement immediately follows. □

Overall, the three defined lottery extensions can be ordered in terms of granularity.

Lemma 1.10
For any preference relation R it holds that $R^{\mathrm{ST}} \subseteq R^{\mathrm{SD}} \subseteq R^{\mathrm{DL}}$.

Proof. We prove the relations separately.

- $R^{\mathrm{SD}} \subseteq R^{\mathrm{DL}}$: Let p and q be lotteries with $p\ R^{\mathrm{SD}}\ q$. Let $E^1, \ldots, E^k$ be the equivalence classes induced by R. If there exists no l such that $p(E^l) \neq q(E^l)$ the proof is finished. Thus assume w.l.o.g. there exists such a l and chose it as the smallest, i.e. $p(E^i) = q(E^i)$ for all $i < l$ and $p(E^l) \neq q(E^l)$. By Corollary 1.9, $p(\bigcup_{i=1}^{l} E^i) \geq q(\bigcup_{i=1}^{l} E^i)$, together one arrives at $p(E^l) > q(E^l)$ and thus $p\ R^{\mathrm{DL}}\ q$.
- $R^{\mathrm{ST}} \subseteq R^{\mathrm{SD}}$: Let p and q be lotteries with $p\ R^{\mathrm{ST}}\ q$. Using the notation from the proof of Proposition 1.7 one has that $a\ P\ b$ and $b\ P\ c$ for all $a \in A_p, b \in A_\cap, c \in A_q$. Assume that there exists an $l \in \mathbb{N}$ such that $\sum_{i=1}^{l} p(E^i) < \sum_{i=1}^{l} q(E^i)$. Define $E = \bigcup_{i=1}^{l}$. Now, distinguish the following cases:
  - $E \cap \widehat{q} = \emptyset$: This immediately yields a contradiction as then $q(E) = 0 \leq p(E)$.
  - $E \cap \widehat{q} \subset \widehat{p}$: By Proposition 1.7, one has that $p(a) = q(a)$ for all $a \in \widehat{p} \cap \widehat{q}$ which gives $q(E) \leq p(E)$.
  - $E \cap \widehat{q} \setminus \widehat{p} \neq \emptyset$: Proposition 1.7 states that for every $a \in \widehat{p}, b \in \widehat{q} \setminus \widehat{p}$ one has $a\ P\ b$. Thus, if $E \cap \widehat{q} \setminus \widehat{p} \neq \emptyset$, the entirety of $\widehat{q}$ already has to be covered by E, i.e. $\widehat{p} \subset E$. Therefore $p(E) = 1 \geq q(E)$. □

SD is often highlighted as one of the most important lottery extensions because of its well known relation to von Neumann-Morgenstern utility functions.

Lemma 1.11
Let R be an arbitrary preference relation over A and let $p, q \in \triangle(A)$. Then $p\ R^{\mathrm{SD}}\ q$ iff for every von Neumann-Morgenstern utility function compatible with R the expected utility of p is greater or equal than the one of q.

The proof is left out here.

Efficiency

Using lottery extensions, efficiency of a correspondence can be defined intuitively. As an intermediate concept, the notion of dominance between single alternatives and lotteries is needed.

Definition 1.12 (Dominated lotteries and alternatives)
Given a vote $\mathfrak{V}$ and a lottery extension χ, a lottery $p \in \triangle(A)$ is called χ-*dominated* (by q w.r.t. $\mathfrak{V}$), if there exists another lottery $q \in \triangle(A)$ such that $q\ {R_i}^{\chi}\ p$ for all agents $i \in N$ and $q\ {P_j}^{\chi}\ p$ for some $j \in N$.

An alternative $a \in A$ is called *Pareto dominated* (by b), if there is an alternative $b \in A$ such that $b\ R_i\ a$ for all $i \in N$ and $b\ P_j\ a$ for some $j \in N$.

Using this definition, it straightforward to arrive at efficiency. In a nutshell, a lottery is efficient, if there exists no other lottery which every agent prefers.

Definition 1.13 (Efficiency)
Let χ be a lottery extension and $\mathfrak{V}$ any vote. A lottery p is called χ-efficient (w.r.t. $\mathfrak{V}$), if there exists no lottery q which χ-dominates p. A correspondence f is called χ-efficient, if for any vote $\mathfrak{V}$ one has that every $p \in f(\mathfrak{V})$ is χ efficient w. r. t. $\mathcal{R}$.

A notion of efficiency which can not be directly expressed in terms of lottery extensions is *Pareto optimality*.

Definition 1.14 (Pareto optimality)
A correspondence f is Pareto optimal (or *ex post efficient*), if for any vote $\mathfrak{V}$ and any $p \in f(\mathfrak{V})$ every Pareto dominated alternative a receives zero probability.

The provided efficiency notations are strongly related to each other, mostly due to the consequences of Lemma 1.10.

Theorem 1.15 ([ABB13a; AS14])
For any lottery p one has that DL-*efficiency* $\Rightarrow$ SD-*efficiency* $\Rightarrow$ *Pareto optimality* $\Rightarrow$ ST-*efficiency.*

Additionally, ST-efficiency is strongly related to the agents' primary preferences.

Corollary 1.16
Given a vote $\mathfrak{V}$, a lottery p is ST*-efficient iff there exists an agent i with $p(E_i^1) > 0$.*

Proof. If $p(E_i^1) > 0$, Corollary 1.8 shows that there exists no lottery q with $q\ P_i^{\text{ST}}\ p$. Together with Proposition 1.7 this shows that $q\ R_i^{\text{ST}}\ p$ implies $p = q$. Hence there exists no q which ST-dominates p.

If on the other hand there exists no agent with $p(E_i^1) > 0$, an uniform distribution over $\bigcup_{i=1}^n E_i^1$ is ST preferred by any agent. □

This directly translates into a sufficient condition for ST-efficiency of a correspondence:

Corollary 1.17
A correspondence f is ST*-efficient, iff for any vote $\mathfrak{V}$ one has that for every $p \in f(\mathfrak{V})$ there exists an agent $i \in N$ with $p(E_i^1) > 0$.*

Manipulability & Strategyproofness

As we previously mentioned, agents are not required to vote according to their preferences. One may think that they potentially could manipulate the result in their favour by deliberately misreporting their preferences. This issue was discovered a long time ago and vast amount of research was put into it, with mentions going as far back as the 18th century. Jean-Charles de Borda himself noted, that his *Borda count* method (which still is a popular procedure, employed in political and academical elections etc.) is manipulable and, after being confronted with this issue, replied that "[his] scheme is only intended for honest men" ([Bla+58, p. 182]). While by virtue of the previously mentioned Gibbard-Satterthwaite theorem, deterministic systems always suffer from the possibility of tactical voting, correspondences can be more or less resilient to it. The definition of manipulability in the domain of randomized voting again relies on lottery extensions, where a correspondences is manipulable, if an agent can achieve a better outcome by altering his reported preference relation.

Definition 1.18 (Manipulability)
Let χ be a lottery extension. A correspondence f is called χ-*manipulable*, if there exists a set of agents N, some alternatives A, an agent $i \in N$ and profiles $\mathcal{R}$, $\mathcal{R}'$ such that $R_{-i} = R'_{-i}$ (i.e. only agent i modified his preferences) where for every $p \in f(\mathfrak{V})$ one has $q\ R_i^{\chi}\ p$ for every $q \in f(\mathfrak{V}')$ and $q\ P_i^{\chi}\ p$ for some q.

Notice that for an essentially single-valued correspondence the above boils down to the following statement: For every $p \in f(\mathfrak{V})$ and $q \in f(\mathfrak{V}')$ one has that $q\ P_i^{\chi}\ p$, i.e. under some circumstances a certain agent can alter his reported preferences to be better off w.r.t. his true preferences.

Definition 1.19 (Strategyproofness)
Let χ be a lottery extension. A correspondence f is called *weakly χ-strategyproof* (weakly χ-SP), if it is not χ-manipulable. It is called *strongly strategyproof*[2] (strongly χ-SP), if for every set of agents N, alternatives A, agent $i \in N$ and preference profiles $\mathcal{R}$, $\mathcal{R}'$ with $R_{-i} = R'_{-i}$ one has that every $p \in f(\mathfrak{V})$ fulfils $p \; R_i{}^\chi \; q$ for every $q \in f(\mathcal{R}')$.

Note that the distinction between strong and weak strategyproofness is only important for lottery extensions which are not complete over the domain of lotteries. Strategyproofness can be explained nicely in game theoretical terms, interpreting the submission of a vote as a player's strategy in the voting game: Weak strategyproofness means that submitting sincere preferences never is a dominated strategy of any agent, strong strategyproofness corresponds to honesty always being a dominant strategy.

Again, strategyproofness induced by different lottery extensions are related to each other due to Lemma 1.10.

Theorem 1.20 ([ABB14; AS14])
For any correspondence f one has that strongly SD-*SP* $\Rightarrow$ DL-*SP* $\Rightarrow$ *weakly* SD-*SP* $\Rightarrow$ *weak* ST-*SP.*

While weak ST-strategyproofness thus arguably is the weakest notion of the presented ones, it "only" allows a manipulator at most to skew the resulting lottery, he can't change the support of it to a different, disjoint one.

Corollary 1.21
A correspondence f is weakly ST-*strategyproof, if for every vote $\mathfrak{V}$ one has that $E_i^1 \cap \widehat{p} \neq \emptyset$ for all $p \in f(\mathfrak{V})$ and $i \in N$.*

Proof. Immediately follows from Corollary 1.8, as any agent i can't possibly ST-improve upon p. □

1.2 Assignment Problems and their Relation to Voting

Another heavily researched topic are so called assignment problems (see for example [BM01; AS98; KM10; Man09; RS92]). As assignment theory is closely related to voting (and can even be modelled as a subset of voting), we will briefly explain it here.

As the name suggests, this topic deals with objects being (optimally) distributed among equally many agents with the agents expressing preferences over these objects. Often mentioned examples are the assignment of jobs to workers or the classical "house allocation problem", where the agents want to allocate houses / rooms. The definitions of the society and preference relations stay the same, there are n objects, n agents and

[2] Depending on the literature, the term "strategyproof" either denotes the strong or the weak version

each reports a preference relation over the objects. For similar reasons as in voting, randomization is used to grant ex ante fairness (and open up many other possibilities). Instead of requiring an assignment procedure to return a single assignment $\tau : N \to A$ (which can be identified with permutation matrices), lotteries over such assignments are returned. These random assignments can intuitively be represented through bistochastic matrices by just adding up each assignment matrix according to the lottery. Using the classical Birkhoff-Von Neumann theorem (cf. [MOA10, Theorem A.2]), which states that every bistochastic matrix is a convex combination of permutation matrices, every bistochastic matrix also corresponds to a certain assignment.

Assignment problems can, with a little work, be transformed into voting problems, which we will explain by providing an example. Suppose agent i prefers object a over all others. This can be interpreted as "vote that i receives object a and all others receive something else". Note that the previously mentioned distinction between strategies and outcomes is important. Agents can't vote for certain assignments (which are the outcomes), but only for all assignments where they receive certain objects. More formally, given an assignment problem with agents N, alternatives A and preference profile $\mathcal{R}$, the corresponding voting problem looks as follows: Define $A' := \{v \in A^n : v(i) \neq v(j) \, \forall \, i, j \in N, i \neq j\}$ as the set of all possible assignments. The agents now vote on each assignment, with the requirement that they have to be indifferent between the other agents assignments: Construct $\mathcal{R}'$ by defining the equivalence classes ${E'}_i^k := \{v \in A' : v(i) \in E_i^k\}$ for all $i \in N, k \leq m_i$. Applying a correspondence to the vote $(N, A', \mathcal{R}')$ results in a set of lotteries over A', which can again be transformed into a classical assignment. Let p be such a lottery, the probability of i receiving a is given by $\sum_{v \in A' : v(i) = a} p(v)$.

1.3 Examples of Social Decision Schemes

As we will use some established decision schemes on various occasions throughout this thesis, we shall introduce them to the reader in a compressed manner. Should the need for further study arise, we recommend the respectively listed literature.

1.3.1 Random Dictatorship

The first decision scheme to be introduced is considered to be one of the most beautiful, yet most limited schemes. First defined by Gibbard [Gib77], *random dictatorship* (RD) basically selects a random agent as dictator and lets him decide. This is repeated for every agent and the results are averaged into a probability distribution, effectively choosing each alternative proportionally to its plurality score.

Definition 1.22 (Random dictator)
Let $\mathfrak{V}$ be a vote with a strict profile $\mathcal{R}$. Random dictator is defined by

$$\mathrm{RD}(\mathfrak{V}) := \sum_{i \in N} \frac{1}{n} E_i^1.$$

Theorem 1.23
Random dictator is DL-efficient and (strongly) SD-strategyproof.

Although Gibbard [Gib77] already mostly proved this statement, we will provide a complete proof to illustrate the concepts of efficiency and strategyproofness.

Proof. Let $\mathcal{R}$ be a strict preference profile, define $p := \mathrm{RD}(\mathcal{R})$.

- **Efficiency:** Suppose there exists a lottery q which DL-dominates p. As $p \neq q$ there is an $a \in A$ with $p(a) > q(a)$. By definition of RD one has $p(\bigcup_{i \in N} E_i^1) = 1$, together with strictness of $\mathcal{R}$ this ensures the existence of an agent i with $E_i^1 = \{a\}$. Hence $q(E_i^1) < p(E_i^1)$, in contradiction to $q \; {R_i}^{\mathrm{DL}} \; p$.

- **Strategyproofness:** Let $\mathcal{R}$ be an arbitrary (strict) preference profile and i any agent. Additionally, modify i's preferences arbitrarily to arrive at $\mathcal{R}'$, i.e. $R_{-i} = R'_{-i}$. Let $p = \mathrm{RD}(\mathfrak{V})$ and $q = \mathrm{RD}(\mathfrak{V}')$. If i's primary preference stays the same, then $p = q$ as only the primary preferences of agents are considered by the procedure. If on the other hand i changes his primary preference from e_i to some e'_i then by definition of RD one has $q(e_i) = p(e_i) - \frac{1}{n}$ and $q(e'_i) = p(e'_i) + \frac{1}{n}$, especially $q(E_i^1) < p(E_i^1)$ and thus $p \; {P_i}^{\mathrm{DL}} \; q$. □

Gibbard also has shown (in the same paper) that RD is indeed the only scheme fulfilling these properties on the strict domain. In the light of Theorems 1.15 and 1.20 this is one of the reasons why RD is considered to be a very powerful solution to strict voting. Aside from these strong properties, there exist further results strengthening random dictator, see for example [AS12].

However these intriguing properties unfortunately are not carried over to the full domain. The naive extension

$$\mathrm{RD}'(\mathcal{R}) := \sum_{i \in N} \frac{1}{n} \left(\sum_{a \in E_i^1} \frac{1}{|E_i^1|} a \right)$$

quickly leads to severe problems. Consider the profile

$$\begin{aligned} 1 &: \{a, b, c\}, \{d\} \\ 2 &: \{a\}, \{b, c, d\} \\ 3 &: \{a\}, \{b, c, d\} \\ 4 &: \{a\}, \{b, c, d\}. \end{aligned}$$

Clearly, $p = 1 \cdot a$ is the best solution to this vote, but the above procedure would return $p = (\frac{3}{4} + \frac{1}{4} \cdot \frac{1}{3})a + \frac{1}{12}b + \frac{1}{12}c$. RD's canonical extension *random serial dictatorship* (RSD) (discussed in e.g. [AS98; BM01; Sve94; ABB13a; ABB13b][3]) considers all possible "chain of commands", sequentially narrowing down the selected set.

Definition 1.24 (Random serial dictator)
Define Π^N as the set of all possible permutations π of N. Let $\pi \in \Pi^N$, then define

$$\sigma(R, \pi) := \max_{R_{\pi(n)}} \left(\max_{R_{\pi(n-1)}} \left(\dots \left(\max_{R_{\pi(1)}} (A) \right) \dots \right) \right).$$

That is, $\sigma(R, \pi)$ is obtained by letting agents sequentially (according to π) pick their most liked alternatives from the set which the previous agent picked. Note that always $|\sigma(R, \pi)| \geq 1$. If $\sigma(R, \pi)$ should contain more than one element, every agent is indifferent between these elements (which rarely occurs).

Let now $\pi_1, \dots, \pi_{n!}$ be an enumeration of Π^N. Define RSD as follows:

$$\mathrm{RSD}(\mathfrak{V}) := \left\{ \sum_{j=1}^{n!} \frac{1}{n!} p_j : p_j \in \triangle(A) \text{ such that } \widehat{p_j} \subset \sigma(R, \pi_j) \right\}.$$

Any sub-correspondence RSD is called RSD *scheme*.

The following properties of RSD are well known:

Theorem 1.25
RSD *is*

- *essentially single-valued,*
- *Pareto-optimal but not* SD-*efficient and*
- *strongly* SD-*strategyproof.*

It is easily verified that RSD is essentially single-valued. Proving its Pareto-optimality is straightforward and will be left out here. SD-strategyproofness can be proven by an analogous argumentation as for RD. But, as previously mentioned, RSD is not as efficient as RD, which we will show by providing a counterexample. Define $\mathcal{R}$ as follows:

$$1 : \{a, b\}, \{c\}, \{d\} \qquad 3 : \{a\}, \{d\}, \{b, c\}$$
$$2 : \{c, d\}, \{a\}, \{b\} \qquad 4 : \{c\}, \{b\}, \{a, d\}.$$

[3] Some of these works apply the idea to assignment problems. Bogomolnaia and Moulin [BM01] name this algorithm *random priority*

RSD yields the unique lottery $\frac{5}{12}a + \frac{1}{12}b + \frac{5}{12}c + \frac{1}{12}d$, which clearly is SD-dominated by $\frac{1}{2}a + \frac{1}{2}c$. There are $4! = 24$ possible permutations of the agents. If the permutation starts with agent 3 (4), clearly a (c) gets selected. If on the other hand the permutation begins with 1 and is followed by 4 (2 followed by 3), then b (d) are selected. In general, RSD apparently seems to struggle with disagreements between the agents. RSD received further criticism because of its computational complexity, the amount of permutations grows in $n!$ and - in the worst case - every permutation has to be evaluated.

1.3.2 Maximal Recursive Rule

Aziz [Azi13] presented an alternative to RSD, named *maximal recursive rule* (MR), which is slightly less strategyproof but more efficient and computable in polynomial time. We will present the algorithm and some of its core properties, for further discussion the defining paper is recommended. MR relies on what Aziz named *inclusion minimal subsets*.

Definition 1.26 (Inclusion minimal subsets)
Let $S \subseteq A$, $A_1, \ldots, A_m$ subsets of S, $I(A_1, \ldots, A_m)$ the set of non-empty intersections of some subset of $\{A_1, \ldots, A_m\}$, i.e.

$$I(A_1, \ldots, A_m) := \left\{ X \in 2^A \setminus \emptyset : X = \bigcap_{A_j \in A'} A_j \text{ for } A' \subset \{A_1, \ldots, A_m\} \right\}.$$

The inclusion minimal subsets of $S \subseteq A$ with respect to $A_1, \ldots, A_m$ are defined as

$$\text{IMS}(A_1, \ldots, A_m) := \{ X \in I(A_1, \ldots, A_m) : \nexists X' \in I(A_1, \ldots, A_m) : X' \subsetneq X \}.$$

Informally, a set S is an element of $\text{IMS}(A_1, \ldots, A_k)$ if S is an intersection of some A_is and is disjoint from all other A_js.

Aziz [Azi13] has shown that these inclusion minimal subsets can be computed in polynomial time by a greedy-type algorithm. Another small definition used is the notion of *generalized plurality score*:

Definition 1.27 (Generalized plurality score)
The generalized plurality score of a according to $\mathcal{R}$ restricted to $S \subset A$ is defined as

$$s^1(a, S, \mathfrak{V}) := \left| \left\{ i \in N : a \in \max_{R_i}(S) \right\} \right|.$$

This gives the amount of agents which consider a to be one of the best alternatives out of S. Using this, maximal recursive rule can be defined as follows:

Algorithm 1: Maximal recursive rule

Input: A, N, $\mathcal{R}$
Output: p

```
Procedure MR-subroutine(S, ν, (A, N, R))
    if max_{R_i}(S) = S for all i ∈ N then
        return ∑_{a∈S} ν/|S| · a
    else
        foreach i ∈ N do
            M(i, S) ← max_{R_i}(S)
            T(i, S) ← arg max_{b∈M(i,S)} s^1(b, S, R)
            foreach a ∈ S do
                if a ∈ T(i, S) then
                    t(i, a) ← 1/|T(i, S)|
                else
                    t(i, a) ← 0
                end
            end
        end
        foreach a ∈ S do
            γ(a) ← ∑_{i∈N} t(i, a)
            p'(a) ← (ν/n) · γ(a)
        end
        {S_1, ..., S_k} ← IMS(M(1, S) ∩ \hat{p'}, ..., M(n, S) ∩ \hat{p'})
        p ← 0
        foreach S' ∈ {S_1, ..., S_k} do
            p ← p + MR-subroutine(S', p'(S'), (A, N, R))
        end
        return p
    end
return MR-subroutine(A, 1, (A, N, R))
```

We will briefly outline the steps of the algorithm. MR recursively (re)distributes probabilities, the recursion routine works as follows: S, a subset of A, indicates the set over which the fraction ν of the total probability weight has to be distributed. If each agent is indifferent between all elements of S, ν is uniformly distributed over S. Otherwise, S is split into subsets. Every agent i first selects the set of his most preferred alternatives $M(i, S)$ in S and then generates $T(i, S)$, a set containing all elements of $M(i, S)$ which are maximal w. r. t. the generalized plurality score over

$M(i, S)$. Based on this, every agent i then uniformly spreads a weight of one over $T(i, S)$, assigning the resulting weight to $t(i, a)$. Summing this contribution over all agents for an alternative a gives the total relative score of a, denoted by $\gamma(a)$. As summing all γs would then yield n instead of the required ν, γ is scaled by $\frac{\nu}{n}$, arriving at p'. As agents might still want to redistribute some weight on subsets of S, the inclusion minimal subsets of each $M(i, S)$ intersected with the support of the intermediate probability distribution p' are computed. The then subroutine is called on every of these subsets, again redistributing its relative probability according to p'. As the result of IMS always is a set of disjoint sets, this does not change the overall weight of any subset, the weight is just redistributed.

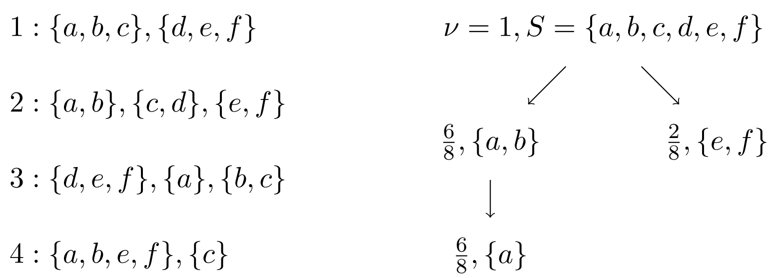

Figure 1.1: Example of MR

Fig. 1.1 shows the conceptualized execution of the MR-algorithm given the specified profile. The resulting lottery is given by $p = \frac{6}{8}a + \frac{1}{8}e + \frac{1}{8}f$, RSD in contrast yields $\frac{16}{24}a + \frac{1}{8}d + \frac{5}{24} \triangle (\{e, f\})$. Aziz [Azi13] has proven the following properties of MR, which will not be reasserted here.

Theorem 1.28 ([Azi13])
MR *is*

- *Pareto-optimal and* DL^1*-efficient* ($q \; R_i^{\mathrm{DL}^1} \; p$ *if* $q(E_i^1) \geq p(E_i^1)$*),*
- DL-*SP, but not strongly* SD-*SP,*
- *equivalent to* RD *on the strict domain,*
- SD-*efficient for some instances where* RSD *is not.*

1.3.3 Strict Maximal Lotteries

Motivated by a more game theoretic view of votes, both Kreweras [Kre65] and Fishburn [Fis84] defined and studied so called *maximal lotteries* independently from each other. Based on this definition *strict maximal lotteries* (SML) have been defined. Additional discussion of SML and similar concepts can be found among others in [LLL93; FR95; RS10], the content of this chapter is primarily relying on the variation presented by Aziz, Brandt, and Brill [ABB13b]. The core idea is based on the so called *majority margin*, which is the amount of voters that prefer one alternative over the other.

Definition 1.29 (Majority margin)
Given a vote $\mathfrak{V}$, the majority margin $g_{\mathfrak{V}}$ of a over b is defined as

$$g_{\mathfrak{V}}(a, b) := |\{i \in N : a \ R_i \ b\}| - |\{i \in N : b \ R_i \ a\}|.$$

A *maximal element* element of the majority margin (often called *Condorcet winner*) is an alternative a with $g_{\mathfrak{V}}(a, b) \geq 0$ for all $b \in A$. One can argue that such a maximal element is a good candidate for being selected by a SDS, the idea of requiring this even goes back as far as the 18th century, where [Car+85] first introduced it alongside his famous Condorcet method. There exists much work on "Condorcet efficiency" in general ([AS12; EFS12; RS10]), but we will not delve further into this topic. It should be noted that such an element often fails to exist. To circumvent this problem, $g_{\mathfrak{V}}$ can be extended to lotteries:

Definition 1.30 (Expected majority margin)
Let $\mathfrak{V}$ be a vote, p and q some lotteries over A. The *expected majority margin* of p over q is defined as

$$g_{\mathfrak{V}}(p, q) := \sum_{(a,b) \in A \times A} p(a) q(b) g_{\mathfrak{V}}(a, b).$$

Using this expected majority margin *maximal lotteries* are defined analogous to maximal elements:

Definition 1.31 (Maximal & strict maximal lotteries)
Maximal lotteries of $\mathfrak{V}$ are given by

$$\mathrm{ML}(\mathfrak{V}) := \{p \in \triangle(A) : g_{\mathfrak{V}}(p, q) \geq 0 \text{ for all } q \in \triangle(A)\}.$$

Strict maximal lotteries are maximal lotteries with inclusion maximal support:

$$\mathrm{SML}(\mathfrak{V}) := \{p \in \mathrm{ML}(\mathfrak{V}) : \widehat{q} \subseteq \widehat{p} \text{ for all } q \in \mathrm{ML}(\mathfrak{V})\}.$$

Similar to non-cooperative games, where the introduction of mixed strategies guarantees existence of Nash equilibria, extending the majority margin to a convex set ensures the presence of maximal lotteries. By interpreting $g_{\mathfrak{V}}$ as the payoff matrix of a symmetric zero-sum game, maximal lotteries even are equivalent to the Nash equilibria of this game and the non-emptiness of ML($\mathfrak{V}$) is actually just a special case of Neumann's minimax theorem ([Neu28]). As ML($\mathfrak{V}$) clearly is convex, SML($\mathfrak{V}$) is non-empty for any vote $\mathfrak{V}$. Aziz, Brandt, and Brill [ABB13b] have shown the following properties of SML:

Theorem 1.32 ([ABB13b])
SML *is*

- SD-*efficient,*
- *not weakly* SD-*SP (even on the strict domain) but weakly* ST-*SP*

Also, SML always selects a Condorcet winner (if one exists). Hence SML is not equivalent to RD on the strict domain. This can be visualized by a simple construction. Let $n \geq 3$, $m \in \mathbb{N}$, $A = \{a, a_1, \dots, a_n, b_1, \dots, b_m\}$. Define $\mathcal{R}$ as follows:

$$\begin{array}{l}
1 : a_1,\ a,\ b_1,\ b_2,\ \dots,\ b_{m-1},\ b_m,\quad a_2,\ a_3,\ \dots,\ a_{n-1},\ a_n \\
2 : a_2,\ a,\ b_2,\ b_3,\ \dots,\ b_m,\quad b_1,\quad a_3,\ a_4,\ \dots,\ a_n,\quad a_1 \\
3 : a_3,\ a,\ b_3,\ b_4,\ \dots,\ b_1,\quad b_2,\quad a_4,\ a_5,\ \dots,\ a_1,\quad a_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
n : a_n,\ a,\ b_m,\ b_1,\ \dots,\ b_{m-2},\ b_{m-1},\ a_1,\ a_2,\ \dots,\ a_{n-2},\ a_{n-1}
\end{array}$$

Here a clearly is a Condorcet winner, every agent prefers a over any b_i and a is preferred over a_i by every but one agent. With this SML($\mathcal{R}$) $= \{1 \cdot a\}$, but RD yields $p = \sum_{i \in N} \frac{1}{n} \cdot a_i$. For growing m and n, it becomes more and more unclear which result actually is better. More generally, this result implies an at first unexpected inherent incompatibility between DL- and Condorcet-efficiency. Note that this especially implies that SML is not DL-Efficient.

1.3.4 Egalitarian Simultaneous Reservation

Introduced by Aziz and Stursberg [AS14], ESR is a relatively new algorithm with some interesting concepts, which our new algorithms rely upon. ESR is motivated by the *probabilistic serial rule* (PS), an algorithm working on the strict assignment domain ([Man09; KM10; BM01]). PS can be explained very intuitively: Every agent continuously "eats" infinitesimal parts of his most liked and still available object until it is consumed entirely and advances to the next available (not completely eaten) object. This is repeated until all objects are eaten. As agents eat with the same speed, this happens exactly when every agent has eaten a mass of 1. The fraction of an object which an agent has eaten corresponds to the probability of him receiving it.

Algorithm 2: Egalitarian simultaneous reservation [AS14]

Input : $\mathfrak{V} = (N, A, \mathcal{R})$: The vote

Output : The set of lotteries $\mathrm{ESR}(\mathfrak{V})$

Symbols: $c : 2^A \to \mathbb{R}$: Current ceiling height of each tower,
$h : N \to \mathbb{R}$: Current height of each agent,
$k : N \to \mathbb{N}$: Number of the equivalence class each agent currently is in

foreach $S \subset A$ **do** $c(S) \leftarrow 0$ `// Initialize tower heights`
foreach $i \in N$ **do** $h(i) \leftarrow 0$, $k(i) \leftarrow 1$ `// Initialize agent data`
$t \leftarrow 0$ `// Current time`
while $t < 1$ **do**
  `// computeLambda determines how high agents can climb until some of them bounce off (`$\lambda^\star$`) and which agents do bounce after climbing (`$N^\star$`). The procedure can be found on Page 51.`
  $(\lambda^\star, N^\star) \leftarrow$ `computeLambda(`N, A, h, c`)`
  foreach $i \in N$ **do** `// Do the climbing`
    $c(E_i^{k(i)}) \leftarrow \max\{c(E_i^{k(i)}), h(i) + \lambda^\star\}$ `// Push the current tower of` i
    if $i \in N^\star$ **then** `//` i `dropped off, move to the next class`
      $h(i) \leftarrow 0$
      $k(i) \leftarrow k(i) + 1$
    else `//` i `can climb without hitting a frozen ceiling`
      $h(i) \leftarrow h(i) + \lambda^\star$
    end
  end
  $t \leftarrow t + \lambda^\star$
end
return *All lotteries* p *with* $p(S) \geq c(S)$ *for all* $S \subset A$

PS has been generalized to the *extended probabilistic serial rule* (EPS) by Katta and Sethuraman [KS06], working on the full assignment domain.

ESR progressively restricts the set of all lotteries $\triangle(A)$ by imposing lower bounds on the overall probability of certain equivalence classes. Each class E is represented (and identified with) a *tower* which has a *ceiling* that can be pushed upwards over the course of time, starting at height 0. This ceiling-height corresponds to the lower bound of this class and is denoted by $\mathrm{ceil}_t(E)$ at time t. In other words, the set of all eligible lotteries at time t is specified by $\{p \in \triangle(A) \mid p(E) \geq \mathrm{ceil}_t(E) \,\forall\, E \subset A\}$. Towers are “frozen” at their current height when the existence of a lottery satisfying all current constraints is threatened by pushing it further. At the beginning of the algorithm every agent moves to his most preferred tower, starting to climb and push the ceiling until the tower freezes. When this happens, every agent touching the ceiling drops to the ground and moves to his next preferred tower, again climbing this tower until they hit a frozen ceiling. In particular agents may climb already frozen towers, dropping down as soon as they reach the ceiling. The execution is finished when all agents visited each tower of their preferences. Note that agents neither cooperate (i.e. they don't push faster if they are climbing the same tower) nor can they “jump up”, they always enter a tower at zero height no matter how high the ceiling currently is. We will provide an example to illustrate how the algorithm works.

Example 1.33
Let the profile $\mathcal{R}$ be given by

$$1 : \{a, b, c\}, \{d\}, \{e\} \qquad 3 : \{d, e\}, \{a, c\}, \{b\}$$
$$2 : \{a, c, d\}, \{b\}, \{e\} \qquad 4 : \{a, b, e\}, \{c, d\}$$

The first freeze happens at $t = \frac{1}{2}$, with the lower bounds being $\text{ceil}_t(\{a, b, c\}) = \text{ceil}_t(\{a, c, d\}) = \text{ceil}_t(\{d, e\}) = \text{ceil}_t(\{a, b, e\}) = \frac{1}{2}$. The towers freezing are $\{a, b, c\}$ and $\{d, e\}$. With both being at height $\frac{1}{2}$, they surely can't grow any more, as there exists no $p \in \triangle(A)$ with $p(\{a, b, c\}) = p(\{d, e\}) = \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$. Agents 1 and 3 drop down and move to tower $\{d\}$ and $\{a, c\}$ respectively. They all continue pushing until $t = \frac{3}{4}$, where $\{d\}$, $\{a, b, d\}$ and $\{a, b, e\}$ freeze. As at least $\frac{1}{2}$ has to go towards $\{a, b, c\}$, $\text{ceil}_t(\{a, b, d\}) = \text{ceil}_t(\{a, b, e\}) = \frac{3}{4}$ leaves only $\frac{1}{4}$ to d and e each. The agents continue their work until the execution is finished with $\text{ESR}(\mathcal{R}) = \{\frac{1}{2}a + \frac{1}{4}d + \frac{1}{4}e\}$.

If not otherwise noted, the following statements and their corresponding proofs can be found in [AS14]. As an immediate consequence of it's definition, one can show that ESR is essentially single-valued.

Theorem 1.34 (Single-valuedness)
ESR *is essentially single-valued.*

Interestingly, the algorithm also is completely oblivious of completely equal agents. If two agents report the same preference relation the result also is the same as if there would have been only one agent reporting this relation.

Corollary 1.35
Let $\mathfrak{V}$ be an arbitrary vote and $i \in N$. Define $R_{n+1} := R_i$ and define (by slight abuse of notation) $\mathcal{R}' := (\mathcal{R}, R_{n+1})$. Then, with $\mathfrak{V}' := (N \cup \{i + 1\}, A, \mathcal{R}')$, one has that $\text{ESR}(\mathfrak{V}) = \text{ESR}(\mathfrak{V}')$.

Proof. Follows directly from the definition of ESR, as agents i and $n + 1$ at all times push the same towers at the same height and thus one bounces off a ceiling iff the other bounces. □

Applying this fact for strict preferences gives a nice characterization of ESR for this domain.

Corollary 1.36
Let $\mathfrak{V}$ be a vote with a strict preference profile $\mathcal{R}$. Define $E := \bigcup_{i \in N} E_i^1$. Then

$$\mathrm{ESR}(\mathcal{R}) = \left\{ \sum_{a \in E} \frac{1}{|E|} a \right\},$$

i.e. ESR *yields the uniform lottery over all alternatives any agents likes the most. Especially,* ESR *is not equivalent to* RD *on the strict domain.*

In spite of that, ESR is DL-efficient.

Theorem 1.37 (Efficiency)
ESR *is* DL-*efficient on the full domain.*

While there are other notions of efficiency incomparable to DL-efficiency ([ABB14]), ESR is among very few other correspondences "surpassing" SD-efficiency. As the proof of DL-efficiency is both short and instructive, it is included here.

Proof. Let $\mathfrak{V}$ be arbitrary and $p \in \mathrm{ESR}(\mathfrak{V})$. Suppose p is DL-dominated by another lottery q for some agent i. There has to exists a time t where some E_i^k froze with $\mathrm{ceil}_t(E_i^k) < q(E_i^k)$. Let w.l.o.g. t be the first time of this happening. As E_i^k was frozen at t, there exists no lottery q' with $q'(E_i^k) > \mathrm{ceil}_t(E_i^k)$ and $q'(E) \geq \mathrm{ceil}_t(E)$ for all other $E \subset A$. □

Unfortunately, ESR's strategyproofness does not match its efficiency.

Theorem 1.38 (Strategyproofness)
For $n \leq 2$, ESR *is strongly* SD-*SP. For $n \geq 3$, one has that* ESR *is*

- DL-*SP but not strongly* SD-*SP on the strict domain and*
- *weakly* ST-*SP on the general domain.*

As DL-strategyproofness on the strict domain was not shown in [AS14], we provide a proof for this particular property.

Proof. Let $\mathcal{R}$ be a strict preference profile and $i \in N$ some arbitrary agent. Choose a such that $E_i^1 = \{a\}$. From Corollary 1.36 one has that ESR($\mathcal{R}$) only depends on the primary preference of each agent. Suppose that i misreports some other primary preference a' in $\mathcal{R}'$. Let $E = \bigcup_{j \in N} E_j^1$ and E' analogous for $\mathcal{R}'$. Distinguish the following cases:

- $a \notin E'$, implying that i was the only agent liking a the most in $\mathcal{R}$. Clearly $p'(a) = 0$.
- $a \in E'$, $a' \in E$: In this case, agent i switched from an alternative that some other agents like to another, which also is liked by some agents. As a consequence $|E| = |E'|$ and thus $p = p'$.
- $a \in E'$, $a' \notin E$: Here, i switched to some alternative that no agent liked beforehand. This means that $|E'| > |E|$, hence $p'(a) < p(a)$.

In any case $p \; R_i^{\text{DL}} \; p'$. □

Another interesting property shown in [AS14] is a categorization of ESR as part of a larger class of correspondences called *serial reservations*. Serial reservations are defined by so called *climbing speed functions*, (piecewise) continuous functions $s_i : [0,1] \to \mathbb{R}_0^+$ with $\int_0^1 s_i(t) \; \mathrm{d}x \geq 1$. The corresponding SR-algorithm is then given by agents climbing with speed $s_i(t)$ at time t. In the case of ESR we had $s_i(t) = 1$ for all i. Note that requiring $\int_0^1 s_i(t) \; \mathrm{d}x = 1$ is more intuitive, but not necessary for well-definedness, the algorithm will stop at t given by $\int_0^t \sum_{i \in N} s_i(x) \; \mathrm{d}x = n$. The main result of this categorization is that serial reservations characterize DL-efficient lotteries ([AS14, Theorem 8]).

Chapter 2

Fairness

Prior to introducing the new idea of proportionality, we will first explain the motivation and ideas behind fairness in voting and highlight some already known concepts.

2.1 What is Fairness in Voting?

Being fair arguably means to value different entities equally. In the context of voting this can be interpreted as respecting the opinion of each agent equally, especially if their preferences collide in any way. The previously mentioned neutrality and anonymity satisfy this in the most primitive way, essentially implying "equal treatment of (complete) equals". They will not be discussed further, as any algorithm not satisfying these is inherently unsuitable for a discussion of fairness. Anonymity and neutrality are the prime examples of properties we will categorize as *(a-priori) fairness*, assuring agents ex ante of a fair treatment among themselves. Interestingly, research did not go much farther in this category. Besides neutrality and anonymity most work related to fairness is focused on strategyproofness or variations of it, residing in the domain of manipulation or rather its prevention. This means they mostly do not imply general fair treatment of the agents themselves but rather reassure them to be honest by sensible treatment of their reported preferences.

The main issue of anonymity and neutrality being the only concepts of fairness is that they often are not sufficient for agents to feel treated in a fair manner. When partaking in a vote, agents may want to be respected equally, no matter if they are exchangeable with others or are unique among the society. The difficulty of analysing this lies in measuring how much an algorithm respects a certain agent based on its results. One could for example argue that the majority rule is fair, in the sense that every agents' vote weighs the same and might make a difference. This alone does not always seem to be satisfactory, an exemplary fairness issue of the majority rule is the mistreatment of persistent minorities. Suppose a city's population is 2/3rd conservative and 1/3rd progressive. When deciding polarising matters by vote, conservatives will always win, even though the progressive oriented ones are a significant part of the population. Hence many people may be disappointed by their apparent lack of influence. One could argue in favour of this by claiming that consistency is important in political decisions -

which indeed is a valid argument - but, on the other hand, this might limit progress of the cities development and make a considerable amount of the population unhappy. This example illustrates that "what is fair" can heavily depend on the context of the vote - whether the ideas we present are important for a certain problem has to be considered individually. For a detailed explanation of some problems with majority rule, we suggested the excellent discussion by Saunders [Sau10].

It should also be noted that in certain cases agents might not even value strategyproofness very highly. Many strategyproofness properties assume a given arbitrary profile and agent, ensuring that in any case reporting his preferences truthfully is not a bad strategy for the given agent. To be applied practically, an agent has to be able to obtain either all other votes or both the preferences of all other agents and being assured that every one of them will report his preferences truthfully before even casting his vote. Both seem to be infeasible in some cases, an algorithm which is not strategyproof might in many cases still encourage agents to report their true preferences if they do not or cannot obtain full knowledge of the "voting game". Gibbard [Gib73, p. 590] briefly mentions this issue, but when talking about strategyproofness most literature just assumes that the manipulator is able to obtain these informations. Additionally, even if the manipulator were able to obtain full knowledge, it still might be computationally hard to compute a manipulation strategy ([CS06; BTT89a; BO91; FHH06]) or even to compute a winner at all ([BTT89b; HHR97]), making manipulation impractical, too. Of course, there surely are many cases in which it is an important property, but with that in mind agents might not care about strategyproofness at all if they can ensure secrecy of their votes: A committee deciding about budget spendings for example may prefer efficiency and fairness over potential manipulation safety.

Restricting oneself to (randomized) assignment problems seems to remedy the mentioned issues. Here it seems pretty straightforward to justify fairness definitions. When multiple agents compete for a single object, splitting it equally between them seems to be the only sensible way to deal with the tie. If, on the other hand an agent likes an object which nobody else likes, it should be given to him no matter how the others rival each other. With assignment problems essentially being a subdomain of general voting problems, one might ask why these ideas cannot be transferred easily. The main difference between the two problems in this context is the much stronger competitiveness of agents in assignment problems, whereas voting has an intuitive notion of cooperation. If an agent prefers object a in an assignment setting he wants to "take it away" from everybody else. Or, in other words, he is completely content with other agents liking any other object. In a voting problem however he rather wants everybody to like the same object. Only if all other agents like his most preferred object he is completely satisfied. As in the previous example, there might exist groups of agents which have very similar preferences. A fair voting system arguably has to value their most liked object higher than other, not so popular alternatives, ultimately increasing each agents payoff as a consequence of being part of a larger group.

2.2 Known Concepts

To get a grasp on the abstract terminology, we will provide some examples of known “fairness” concepts. A well known and commonly used property is *monotonicity*[1] ([Fis82a; AS14; BO91; CS06; RS10]).

Definition 2.1 (Reinforcing profile)
Given a vote $\mathfrak{V}$, a profile $\mathcal{R}'$ *reinforces* an $a \in A$ (for some agent $i \in N$) compared to $\mathcal{R}$, if $R'_{-i} = R_{-i}$, $a\ R_i\ b \Rightarrow a\ R'_i\ \forall b \in A$ and $b\ R_i\ b' \Leftrightarrow b\ R'_i\ b'\ \forall b, b' \neq a$. Informally, a certain alternative a gets reinforced if the profile $\mathcal{R}$ is changed such that everything stays the same except that one agent values a more.

Definition 2.2 (Monotonicity)
A correspondence f is called monotonic, if for all votes $\mathfrak{V}$ one has that for every $p \in f(\mathfrak{V})$ there exists a $p' \in f(\mathfrak{V}')$ with $p'(a) \geq p(a)$ whenever $\mathcal{R}'$ reinforces a compared to $\mathcal{R}$.

Monotonicity ensures the agents that ranking a single object higher will not result in a loss of probability of that object. We show or disprove monotonicity for the previously introduced algorithms.

Theorem 2.3
RSD *(and hence* RD*),* MR *and* ESR *are monotonic,* SML *is not, even on the strict domain.*

Proof. Aziz [Azi13] shows that MR is monotonic. The (strict) profiles used in [ABB13b, Proposition 7] an example violating the monotonicity of SML. Aziz and Stursberg [AS14] did not provide a proof of ESR’s monotonicity because of space constraints. It is indeed rather technical and thus moved to the appendix on Page 47.

To prove RSD’s monotonicity, let $\mathcal{R}$ and $\mathcal{R}'$ be profiles where $\mathcal{R}'$ reinforces the alternative a for agent i compared to $\mathcal{R}$. Choose an arbitrary agent-permutation $\pi \in \Pi^N$. One has that $a \in \sigma(\mathcal{R}, \pi) \Rightarrow a \in \sigma(\mathcal{R}', \pi)$. Hence for every $p \in \triangle(A)$ with $\widehat{p} \subseteq \sigma(\mathcal{R}, \pi)$ we have that $p' = 1 \cdot a$ satisfies $\widehat{p'} \subseteq \sigma(\mathcal{R}', \pi)$ and clearly $p'(a) \geq p(a)$. This yields the monotonicity of RSD and implies that RD is monotonic, too. □

In a way, monotonicity also encourages agents to report their proper preferences. Suppose an agent likes object a over b, i.e. $a\ P\ b$. If instead he reports $b\ R'\ a$, monotonicity implies that $p(a) \geq p'(a)$ (as $\mathcal{R}$ reinforces a compared to $\mathcal{R}'$), meaning that he cannot increase the probability of obtaining a this way. It may still be the case that some other objects of the agent might still gain probability to yield in a better result. In general, monotonicity is not sufficient for strategyproofness but indeed is required.

[1] Strongly related to *nonnegative response*, as defined by Satterthwaite [Sat75]

Theorem 2.4
DL-*strategyproofness implies Monotonicity.*

Proof. Let f be a DL-strategyproof SDS. Let $\mathfrak{V}$ be an arbitrary vote and $\mathcal{R}'$ a preference profile such that $\mathcal{R}$ reinforces an $a \in A$ compared to $\mathcal{R}'$ for some agent $i \in N$. To show that f is monotonic, one needs to prove that for every $p' \in f(\mathfrak{V}')$ there exists a $p \in f(\mathfrak{V})$ with $p(a) \geq p'(a)$ (Note that $\mathfrak{V}$ and $\mathfrak{V}'$ have opposite roles compared to the definition).

Choose $k \in \mathbb{N}$, $E \subset A$ (possibly $E = \emptyset$) such that $E_i^k = E \cup \{a\}$. Note that (if $E \neq \emptyset$) ${E'}_i^k = E$ by the reinforcing nature of $\mathcal{R}$ (compared to $\mathcal{R}$, a is moved "backward" in $\mathcal{R}'$). As f is DL-SP, misreporting $\mathcal{R}'$ instead of $\mathcal{R}$ and vice versa does not pay off (in the DL sense), i.e. $p\ {R_i}^{\mathrm{DL}}\ p'$ and $p'\ {R'}_i^{\mathrm{DL}}\ p$. As $\mathcal{R}$ and $\mathcal{R}'$ only differ in i's rating of a, one has that all equivalence classes are the same except the two where a was either taken from or put into.

If $p\ {P_i}^{\mathrm{DL}}\ p'$, there exists $j \in \mathbb{N}$ such that $p(E_i^l) = p'(E_i^l)\,\forall\, l < j$ and $p(E_i^j) > p'(E_i^j)$. $p'\ {P'}_i^{\mathrm{DL}}\ p$ analogously gives j'. One shows that $p(E) + p(a) = p(E_i^k) \geq p'(E_i^k) = p'(E) + p'(a)$ and $p'(E) = p'({E'}_i^k) \geq p({E'}_i^k) = p(E)$. There are four cases to distinguish:

- $p\ {P_i}^{\mathrm{DL}}\ p'$, $p'\ {P'}_i^{\mathrm{DL}}\ p$: One proves that $j, j' \geq k$ by contradiction. Suppose that $j < k$, i.e. $p(E_i^j) > p'(E_i^j)$. Assume further that $j \leq j'$, then $p({E'}_i^j) \leq p'({E'}_i^j)$, too ("$<$" if $j' = j$, "$=$" if $j' > j$). Using $E_i^l = {E'}_i^l\,\forall\, l < k$ yields the contraction. Argumentation against $j' < m$ is completely analogous. Note that the case $j < k$ and $j' < j$ is covered by this argumentation, too. The inequalities follow directly.
- $p\ {P_i}^{\mathrm{DL}}\ p'$, $p'\ {I'}_i^{\mathrm{DL}}\ p$: Repeating above argumentation gives $j \geq k$ (as $p'\ {I'}_i^{\mathrm{DL}}\ p$ requires $p({E'}_i^j) = p'({E'}_i^j)$). This results in the first inequality, the second one is a direct consequence of indifference.
- $p\ I^{\mathrm{DL}}\ p'$, $p'\ {P'}_i^{\mathrm{DL}}\ p$: Analogously, one arrives at $j' \geq k$ and the required inequalities.
- $p\ I^{\mathrm{DL}}\ p'$, $p'\ {I'}_i^{\mathrm{DL}}\ p$: The inequalities are a trivial consequence of the indifference.

Putting everything together gives $p(a) \geq p'(a)$. □

Hence monotonicity (somewhat) encourages agents to truthfully report their preferences, as explained above, but it does not a-priori ensure fair treatment of an agent in the sense of his preferences always being valued: Albeit being monotonic, majoritarian rule does not guarantee agents to "win" the vote with at least a minor probability. Because of that, an agent might generally dislike such a scheme, as he feels not treated fairly. Even if the scheme is monotonic, some agents might always receive the worst possible utility (everything but their least-preferred alternative may have zero probability), no matter which preferences they actually report.

The idea of fair outcome share ([AS14][2]) is the first example of a true a-priori fairness property:

Definition 2.5 (Fair outcome share)
Given a vote $\mathfrak{V}$, a lottery $p \in \triangle(A)$ fulfils *fair outcome share*, if for every agent $i \in N$ one has $p(E_i^1) \geq \frac{1}{n}$. A correspondence f fulfils fair outcome share, if for every vote $\mathfrak{V}$ every $p \in f(\mathfrak{V})$ fulfils fair outcome share for this particular vote.

This ensures that agents always have at least a tiny chance of one of their most liked alternatives being selected. In general $\frac{1}{n}$ even is the highest chance that can be granted beforehand, this can be validated quickly by considering profiles with disjoint primary preference sets (i.e. complete disagreement). To demonstrate the concept of fair outcome share, we show that ESR fulfils this property.

Corollary 2.6
ESR *fulfils fair outcome share.*

Proof. Let $\mathfrak{V}$ be an arbitrary vote. If no tower freezes before $t = \frac{1}{n}$ then for every agent i clearly $p(E_i^1) \geq \frac{1}{n}$ with $p \in \text{ESR}(\mathfrak{V})$. Choose e_i such that $e_i \in E_i^1$ for every agent $i \in N$, then $p := \sum \frac{1}{n} e_i$ fulfils $p(E_i^1) > \text{ceil}_t \left(E_i^1\right) = t$ for every $i \in N$ and $t < \frac{1}{n}$. As every agent pushes the tower corresponding to his primary preference at first, it follows that the first freeze cannot occur prior to $t = \frac{1}{n}$, as p satisfies all constraints imposed up to then. Hence, any $p \in \text{ESR}(\mathfrak{V})$ fulfils $p(E_i^1) \geq \frac{1}{n}$. □

Interestingly, fair outcome share also implies a weak notion of efficiency and strategyproofness.

Theorem 2.7
Any correspondence which fulfils fair outcome share is ST*-efficient and weakly* ST*-strategyproof.*

Proof. Let f be a correspondence fulfilling fair outcome share. By assumption, for every vote $\mathfrak{V}$, $p \in f(\mathfrak{V})$ and $i \in N$, one has that $p(E_i^1) \geq \frac{1}{n} > 0$. Corollaries 1.17 and 1.21 yield the result. □

However, fair outcome share does not capture any kind of previously mentioned group dynamics. This becomes apparent by considering the origin of this property: As mentioned, it was first defined for assignment problems, where cooperation is barely existent, and then canonically extended to the full domain without adjustments. In

[2] Originally defined for assignment cake cutting problems under the name of *proportionality* ([BT96]), this idea was introduced to voting by Bogomolnaia, Moulin, and Stong [BMS05] for dichotomous preferences, named "fair share". Aziz and Stursberg [AS14] - to our knowledge - first used this property on the full domain.

an assignment problem where all agents like a certain object the most, each can at most receive $\frac{1}{n}$ of this object, in contrast to voting where unanimity leads to each agent "receiving" the same object fully.

As an example, suppose a city has a referendum on a certain matter where all citizens can vote for either a or b. Depending on the nature of the decision, implementing fair outcome share might be plausible. If now all citizens except one vote for alternative a the lottery $p = \frac{1}{2}a + \frac{1}{2}b$ would be in accordance with fair outcome share. In this case the majority may feel treated unjust with the single citizen having an over-proportional influence on the outcome. To solve this problem, we extend the idea of fair outcome share to groups of agents, which will be presented in the following section.

2.3 Proportionality

Definition 2.8 (Supporting Agents)
Let the vote $\mathfrak{V} = (N, A, \mathcal{R})$ be given, $E \subset A$ and $m \in \mathbb{N}$ arbitrary. The set of *m-supporting agents of E* are defined as $S(E, m, \mathfrak{V}) := \{i \in N \mid \bigcup_{k=1}^{m} E_i^k \subset E\}$ and $s(E, m, \mathfrak{V}) := |S(E, m, \mathfrak{V})|$. This means $S(E, m, \mathfrak{V})$ contains all agents for which the union of their m most preferred equivalence classes is a subset of E. For convenience, we define some abbreviations:

- $S(o, m, \mathfrak{V}) := S(\{o\}, m, \mathfrak{V})$ for $o \in A$,
- $S(E, \mathfrak{V}) := S(E, 1, \mathfrak{V})$ and,
- if it is clear which vote is referenced, the last argument $\mathfrak{V}$ to S may be left out,

analogously for s.

Lemma 2.9
For fixed $E \subset A$, $s(E, m)$ is decreasing in m.

Proof. Let $E \subset A$, $m \in \mathbb{N}$. For every agent $i \in S(E, m+1)$ one clearly has that $i \in S(E, m)$ and thus $s(E, m+1) \leq s(E, m)$. □

Using this definition of supporting agents, we define the following new fairness criterion:

Definition 2.10 (Proportionality)
p is called *weakly proportional w.r.t.* $\mathfrak{V}$, if for all agents i it holds that $p(E_i^1) \geq s(E_i^1)/n$. p is called *strongly proportional*, if for all $E \subset A$ one has $p(E) \geq s(E)/n$. A correspondence f is called weakly / strongly proportional, if for any vote $\mathfrak{V}$ every $p \in f(\mathfrak{V})$ is weakly / strongly proportional (w.r.t. to $\mathfrak{V}$).

Before starting to work with this new definition, we shall prove some basic properties. At first, some direct consequences of the definition are stated.

Corollary 2.11
Strong proportionality implies weak proportionality.

Corollary 2.12
If a correspondence f fulfils weak / strong proportionality, then every sub-correspondence of f does that, too.

Proof. Both statements immediately follow from the definition of proportionality. □

The first non-trivial property is given by the following existence theorem:

Theorem 2.13 (Existence of proportional lotteries)
For every vote $\mathfrak{V}$ there exists a strongly proportional lottery and thus a weakly proportional one, too. This lottery is in general not uniquely defined.

Proof. Let $\mathfrak{V}$ be an arbitrary vote. Chose a vector of alternatives $c \in A^n$ such that $c_i \in E_i^1$ for each agent $i \in N$. Define $p := \sum_{i \in N} \frac{1}{n} c_i$. Then, for every $E \subset A$ one has

$$p(E) = \sum_{c_i \in E} \frac{1}{n} = \frac{|c_i \in E|}{n} \geq \frac{s(E)}{n}, \tag{2.1}$$

as $i \in S(E)$ implies that $c_i \in E_i^1 \subset E$. Thus p is strongly proportional. By Corollary 2.11, this p also is weakly proportional. □

As proportionality was motivated by generalizing fair outcome share, it is not surprising that fair outcome share is implied by proportionality.

Corollary 2.14
Weak proportionality implies fair outcome share.

Proof. Let p be weakly proportional w.r.t. $\mathcal{R}$. Chose $i \in N$ arbitrary. By assumption, one has $p(E_i^1) \geq \frac{1}{n} s(E_i^1)$. Surely $s(E_i^1) \geq 1$, as $i \in S(E_i^1)$, therefore $p(E_i^1) \geq \frac{1}{n}$. □

As previously mentioned, in contrast to fair outcome share group dynamics are somewhat captured by proportionality. When completely competing groups of agents emerge, strong proportionality ensures each group their corresponding fraction of the whole probability.

Theorem 2.15
Let p be strongly proportional w.r.t. $\mathfrak{V}$. Suppose there exists a partition $\{I_k\}_{k=1}^j$ of N such that $\bigcup_{i \in I_k} E_i^1 \cap \bigcup_{i \in I_l} E_i^1 = \emptyset$ for all $k \neq l$. Informally, this means that the set of agents can be partitioned into groups of purely competitive (primary) desires. Then, for every group I_k it holds that $p(\bigcup_{i \in I_k} E_i^1) = \frac{|I_k|}{n}$.

Proof. Let $\{I_k\}_{k=1}^{j}$ be such a partition. Define $E_k := \bigcup_{i \in I_k} E_i^1$ for every $1 \leq k \leq j$. Strong proportionality requires for every k that $p(E_k) \geq \frac{s(E_k)}{n}$. By assumption one has $S(E_k) = I_k$ and thus $p(E_k) \geq \frac{|I_k|}{n}$. But strong proportionality also requires that $p(A \setminus E_k) \geq \frac{s(A \setminus E_k)}{n} = 1 - \frac{|I_k|}{n}$. Again, by assumption an agent supports $A \setminus E_k$ iff he is not in I_k. Together, this yields $p(E_k) = \frac{|I_k|}{n}$. □

With small modifications, a similar statement can be proven for weak proportionality: When each of the competing groups has a "leader", i.e. an agent whose primary preferences equal the groups aggregated primary preferences, the same result holds. A special case of this theorem using the singleton partition $\{N\}$ is given by the following corollary:

Corollary 2.16
If p is a strongly proportional lottery w.r.t. $\mathfrak{V}$, then $p(\bigcup_{i \in N} E_i^1) = 1$, or in other words $\widehat{p} \subset \bigcup_{i \in N} E_i^1$.

Proof. Follows directly from the definition, as $s(\bigcup_{i=1}^{n} E_i^1) = n$. □

The strong ties to the agents primary preferences imply the following inheritance of weak proportionality when a lottery is DL-dominated.

Corollary 2.17
Let $\mathfrak{V}$ be an arbitrary vote. If p is weakly proportional and q DL-dominates p, then q is weakly proportional, too.

Proof. For an arbitrary agent i one has $q(E_i^1) \geq p(E_i^1)$ by DL-dominance and $p(E_i^1) \geq s(E_i^1)/n$ by weak proportionality of p, together this yields $q(E_i^1) \geq s(E_i^1)/n$. □

The analogous statement for strong proportionality does not hold in general. Consider the following profile:

$$1 : \{a, b\}, \{c, d\} \qquad 3 : \{a, d\}, \{c, b\}$$
$$2 : \{a, c\}, \{b, d\} \qquad 4 : \{e\}, \{a, b, c, d\}$$

The lottery $p = \frac{1}{4}b + \frac{1}{4}c + \frac{1}{4}d + \frac{1}{4}e$ is strongly proportional and is DL-dominated by $q = \frac{1}{2}a + \frac{1}{2}e$ (every agent prefers q over p), but $\frac{1}{n}s(\{a, b, c, d\}) = \frac{3}{4} > q(\{a, b, c, d\}) = \frac{1}{2}$.

An other helpful property is the following simplified characterization in the case of essentially single-valued correspondences:

Theorem 2.18
If a correspondence f is essentially single-valued, it is weakly proportional iff for every vote $\mathfrak{V}$ there exists a $p \in f(\mathfrak{V})$ which is weakly proportional.

Proof. $\Rightarrow$: Clear
$\Leftarrow$: Let $\mathfrak{V}$ be arbitrary and $p, q \in f(\mathfrak{V})$ where p is weakly proportional. By essential single-valuedness of f one has that $p(E_i^1) = q(E_i^1)$ and thus $q(E_i^1) \geq s(E_i^1)/n$. □

Again, the equivalent statement for strong proportionality does not hold generally. Consider the profile

$$1 : \{a, b\}, \{c, d\} \qquad 3 : \{c, d\}, \{a, b\}$$
$$2 : \{b, c\}, \{a, d\}$$

Here, $p = \frac{1}{4}a + \frac{1}{4}b + \frac{1}{4}c + \frac{1}{4}d$ fulfils strong proportionality, whereas $p' = \frac{1}{2}b + \frac{1}{2}d$ does not ($p'(\{a, b, c\}) = \frac{1}{2} < \frac{2}{3}$). However, $p(E_i^j) = p'(E_i^j) = \frac{1}{2}$ for all $i \in N, j \in \{1, 2\}$.

2.3.1 Proportionality on the Strict Domain

Limited to the strict domain, proportionality is simplified a lot. Basically it is required that every agent contributes exactly $\frac{1}{n}$ to his most preferred alternative. With this, proportionality interestingly characterizes random dictatorship.

Theorem 2.19 (Proportionality characterizes random dictator)
For a given strict preference profile $\mathcal{R}$, there exists exactly one weakly proportional lottery p. This p is the result of random dictatorship and hence DL*-efficient.*

Proof. Let $\mathcal{R}$ be a strict preference profile. Weak proportionality requires

$$p(e_i^1) \geq \frac{1}{n} s(E_i^1, 1) = \frac{1}{n} \left| \left\{ k \in N : E_k^1 = E_i^1 \right\} \right|.$$

The unique p fulfilling this is given by

$$p = \sum_{a \in A} \frac{s(a)}{n} \cdot a.$$

RD is defined as $\text{RD}(\mathcal{R}) = \sum_{i \in N} \frac{1}{n} E_i^1$, yielding the same p. By Theorem 1.23 p is DL-efficient. □

This has some immediate consequences:

Corollary 2.20
Given a strict preference profile $\mathcal{R}$, a lottery p is strongly proportional iff it is weakly proportional (w.r.t. the $\mathfrak{V}$).

Proof. We have shown that every strongly proportional lottery is also weakly proportional and that a strongly proportional lottery always exists. Hence, uniqueness of the weakly proportional lottery implies uniqueness of (and equality to) the strongly proportional one. □

Corollary 2.21
Every proportional correspondence is essentially single-valued, DL*-efficienty, strongly* SD-*SP and monotonic on the strict domain.*

Proof. Let f be a proportional correspondence. By Theorem 2.19, f is equivalent to RD, this immediately proves the statement. □

2.3.2 Justification and Flaws of Proportionality

As with any new definition, the soundness and applicability of proportionality have to be exposed. Theorem 2.19 already provides a striking argument for proportionality, as random dictatorship is, as already mentioned, considered to be one of the best SDS existing on the strict domain. However when dealing with the full domain, this definition does indeed seem to have some flaws, of which some will be highlighted and partially refuted in this section.

Firstly, one might argue that proportionality is too "top heavy", focusing only on the primary preferences of agents. This concern can partially be remedied by the following argument: Given complete unanimity, proportionality should of course not require any other outcome than the unanimously supported ones to be selected, no matter what the agents support in their "later" equivalence classes. This demonstrates the relative importance of primary preferences and that they have to be considered at all times. To fix this, one might be encouraged to consider a mixture of equivalence class levels, for example not only the primary preferences, but also unions of the first m equivalence classes. This seemingly tightens the constraints, the basic idea being to consider sums involving m-supporting agents.

Definition 2.22 (Proportionality II)
p is strongly (weakly) proportional II w.r.t. a vote $\mathfrak{V}$ if for all $E \subset A$ ($E \in \{E_i^1 \mid i \in N\}$) and $m \in \mathbb{N}$ it holds that $p(E) \geq s(E, m)/n$.

But a quick proof shows the futility of this definition:

Corollary 2.23
Proportionality II $\Leftrightarrow$ *Proportionality*

Proof. $\Rightarrow$: Clear
$\Leftarrow$: Let p be strongly (weakly) proportional, $E \subset A$ ($E \in \{E_i^1 \mid i \in N\}$), $m \in \mathbb{N}$. By proportionality of p and monotonicity of s: $p(E) \geq s(E, 1)/n \geq s(E, m)/n$. □

There might be a more sophisticated way to incorporate more of the agents' preference relations, but the extreme cases of complete unanimity and complete disagreement should always lead to the same constraints implied by this definition. More generally, the statement of Theorem 2.15 should be implied by any reasonable kind of "strong" proportionality. A drawback of this requirement can be seen by applying it to the profile used to show that SML is not equivalent to RD on Page 18. But as mentioned there, this leads to a more general problem, in a context where RD is accepted as the best SDS on the strict domain the previous requirement seems reasonable. Finding a sort of "bridge" between the different schools of thought on this may provide further insights in this problem, too.

Another prevalent issue one could think of is that agents might inherently be able to manipulate the result. This particular becomes apparent for preference profiles where sets of agents mostly agree with each others primary preferences, but just not quite. Consider, for example, the following preference profile:

$$1 : \{a, b\}, \{c, d\}, \{e\} \qquad 3 : \{a, d\}, \{b, c, e\}$$
$$2 : \{a, c\}, \{b, d, e\} \qquad 4 : \{e\}, \{a, b, c, d\}$$

(Weak) Proportionality grants the group of agents $1, 2, 3$ no "advantage" over 4, although the former have more in common, $p = \frac{1}{4}a + \frac{3}{4}e$ is in accordance with weak proportionality whereas $p' = \frac{3}{4}a + \frac{1}{4}e$ seems to be much fairer. Based on this, one might suspect that agents could potentially be able to manipulate the outcome in their favour by misreporting their preferences in order to join or unite a group of other agents (in the sense of Theorem 2.15). In the above example, agent 1 might misreport $R'_1 : \{a, b, c, d\}$. This guarantees $\frac{3}{4}$ probability for $\{a, b, c, d\}$, which certainly is a SD-improvement for agent 1 compared to p. However, while some proportional algorithms indeed only fulfil weak ST-strategyproofness, there exist algorithms which are both highly strategyproof and proportional.

Theorem 2.24
There exists a strongly SD-strategyproof and strongly proportional SDS on the full domain.

Proof. For a given profile $\mathcal{R}$, iterate over every agent and uniformly distribute $\frac{1}{n}$ weight over his primary preference, i.e.

$$p(a) := \frac{1}{n} \sum_{i \in N : a \in E_i^1} \frac{1}{|E_i^1|}.$$

In other words, for every $E \subset A$ one has

$$p(E) = \sum_{i \in N} \frac{1}{n} \sum_{a \in E_i^1 \cap E} \frac{1}{|E_i^1|} = \frac{1}{n} \sum_{i \in N} \frac{|E_i^1 \cap E|}{|E|}.$$

Every agent is encouraged to report his primary preference accordingly. Suppose an agent i reports E' instead of E as his primary preference. Then

$$p'(E) = p(E) - \underbrace{\frac{1}{n}}_{\text{Agent } i\text{'s previous contribution to } p(E)} + \frac{1}{n} \frac{|E \cap E'|}{|E'|} \leq p(E).$$

Proportionality of the corresponding SDS f can easily be verified. Let $E \subset A$ arbitrary. As shown,

$$p(E) = \frac{1}{n} \sum_{i \in N} \frac{|E_i^1 \cap E|}{|E|} \geq \frac{1}{n} \sum_{i \in N : E_i^1 \subset E} \frac{|E \cap E_i^1|}{|E_i^1|} = \frac{1}{n} \sum_{i \in S(E)} 1 = \frac{s(E)}{n}. \qquad \square$$

It is worth noting that the above defined algorithm is not SD-efficient, though. Answering the question of a proportional, SD-efficient and strongly SD-strategyproof algorithm deems to be difficult. A positive answer would imply finding a both SD-efficient and strongly SD-strategyproof algorithm first. This alone seems to be hard or even impossible to accomplish. Much research was done on this question but only gave negative answers so far ([Zho90; BMS05; ABB14]). Proofs and counterexamples of proportionality for algorithms with various levels of efficiency and strategyproofness suggest that proportionality is somewhat independent of these properties.

A certain concern still arises from the previous example, namely that proportionality does not generate much constraints when the agents' preference classes are overlapping but not nested (in the sense of primary preferences being subsets of each other). This indeed is a weakness of proportionality and improving on that remains an open problem. Instead of considering nesting, one might try to consider intersections, where an agent is said to support a set $E \subset A$ if his primary preferences merely intersect the set. As the amount of intersection highly varies between preference profiles, the intersection score has to be balanced properly, but a much more severe problem quickly unfolds. An agent with primary preference $\{a, b, c\}$ forces non-zero probabilities on a, b and c instead of only requiring the sum of probabilities to be non-negative. This can be used to construct a direct counterexample to efficiency. Let $\mathcal{R}$ be given by

$$1 : \{a, b, c\}, \{d\} \qquad\qquad 2 : \{c, d\}, \{a, b\},$$

the unique DL-efficient lottery to this profile is given by $p = 1 \cdot c$ whereas a naive "intersection score" would require $p' = \frac{1}{5}a + \frac{1}{5}b + \frac{2}{5}c + \frac{1}{5}d$. To fix this issue whilst sticking to the idea of intersections, the intersection score of a set E could for example be based on the amount of primary preference classes that can be intersected such that the intersection is nonempty and a subset of E. For each $E \subset A$, define

$$i(E) := \left| \left\{ I \subset N : \bigcap_{i \in I} E_i^1 \neq \emptyset \wedge \bigcap_{i \in I} E_i^1 \subset E \right\} \right|$$

and put some constraints on E according to this score in a careful manner.

Another idea one could come up with is the following: For a given vote $\mathfrak{V}$, define $\tilde{s}(a)$ as the amount of agents which like a the most, i.e.

$$\tilde{s}(a) := \left| \left\{ i \in N : a \in E_i^1 \right\} \right|,$$

and the summed weight $\tilde{s} := \sum_{a \in A} \tilde{s}(a)$. With that, require for every a that $p(a) \geq \tilde{s}(a)/\tilde{s}$. This already defines p uniquely, in a both proportional and strategyproof way, but violates efficiency: The profile

$$1 : \{a, b\}, \{c\} \qquad\qquad 2 : \{b, c\}, \{a\}$$

would - with this definition - require $p = \frac{1}{4}a + \frac{2}{4}b + \frac{1}{4}c$, whereas $p = 1 \cdot b$ is strictly preferred by both agents.

A final idea to solve the problem of intertwined preferences are Pareto-optimal sets (a set being a subset of every agent's primary preference). They seem to be worthwhile candidates for determining proportionality constraints, as they describe what a group of agents completely agrees upon. So one might search for inclusion-maximal groups of agents having a common Pareto-optimal set of alternatives and require this subset to perform sufficiently well. These sets are generally neither unique nor disjoint, though. As an example, consider the following profile:

$$\begin{aligned} &1 : \{b, a, d\}, \dots && 4 : \{b, c, e\}, \dots \\ &2 : \{a, b, e\}, \dots && 5 : \{c, e, d\}, \dots \\ &3 : \{a, c, d\}, \dots \end{aligned}$$

Here, many 3-tuples of agents have a common Pareto-optimal set, e.g. 1, 2 and 3 have a in common, 3, 4 and 5 have c. However, as there are so many equally sized groups of agents preferring different alternatives, an extension of proportionality defined this way hardly can put up any sensible constraints in this case.

If and how the concept of proportionality can be extended to cover these cases remains an open question. Strong proportionality seems to be a bit too restrictive, some attractive properties of weak proportionality are not preserved, e.g. Corollary 2.17 and Theorem 2.18. Weak proportionality on the other hand appears to be a bit too lax, Theorem 2.15 only holds in special cases. We think that some criterion "in between" may yield a better solution.

2.3.3 Application to known SDS

Now, we will show / disprove proportionality for well known SDS.

Theorem 2.25
MR *and* RSD *satisfy strong proportionality.* SML *and* ESR *on the other hand do not fulfil weak proportionality (even on the strict domain).*

Proof. RSD: Let $\mathfrak{V}$ be arbitrary and p the result of a RSD-Scheme under this preference profile. Denote Π^N as the set of all possible permutations of N. Choose any $E \subset A$. By definition of RSD one has

$$p(E) = \frac{1}{n!} \sum_{\pi \in \Pi^N} p_\pi(E),$$

where p_π is the probability distribution obtained by invoking serial dictatorship in order of π. Out of all this $|\Pi^N| = n!$ possible permutations, there are $s(E) \cdot (n-1)!$ permutations π with $\pi(1) \in S(E)$. For any of these π one has $\operatorname{supp} p_\pi \subset E$ (i.e. $p_\pi(E) = 1$), as for every agent i in $S(E)$ by definition $\max_{R_{\pi(1)}}(A) = \max_{R_i}(A) = E_i^1 \subset E$. As a result, the resulting probability distribution has

$$p(E) \geq \frac{1}{n!} \sum_{\{\pi \in \Pi^N : \pi(1) \in E\}} 1 = \frac{1}{n!} \left| \left\{ \pi \in \Pi^N : \pi(1) \in S(E) \right\} \right| = \frac{1}{n!} s(E)(n-1)! = \frac{s(E)}{n},$$

which concludes the proof.

MR: Let $\mathfrak{V}$ and $E \subset A$ again be arbitrary. In the first step of MR, i.e. running the MR-subroutine with $(A, 1, \mathfrak{V})$, one has for every agent i supporting E that $T(i, A) \subset E$. This follows directly, as only $a \in \max_{R_i(A)} \subset E$ are considered during the computation of $T(i, A)$. Therefore:

$$\sum_{a \in E} p(a) = 1 \cdot \sum_{a \in E} \frac{1}{n} \gamma(a) = \sum_{a \in E} \frac{1}{n} \sum_{i \in N} t(i, a) \geq \sum_{a \in E} \frac{1}{n} \sum_{S(E)} 1 = \frac{s(E)}{n}.$$

In subsequent calls of MR-subroutine, this sum gets redistributed over subsets of E but stays the same overall. Thus $\mathrm{MR}(\mathfrak{V})(E) \geq \frac{s(E)}{n}$.

SML and ESR: As shown on Page 18 (SML) and by Corollary 1.36 (ESR), both algorithms are not equal to RD on the strict domain. Hence by virtue of Theorem 2.19 both also do not fulfil weak proportionality. □

Chapter 3

Proportional SR-based Mechanisms

As shown in Theorem 2.25, ESR does not fulfil proportionality but performs very well in other areas. Inspired by the reservation concept of ESR, we introduce several proportional algorithms.

3.1 Proportional SR

The non-proportionality of ESR stems from the climbing speed function not adapting to cooperation. In order to fix this the speed functions only have to be adapted to reserve the probability that proportionality requires.

Definition 3.1 (Proportional serial reservation)
For a given vote $\mathfrak{V}$, define the function $s_{\mathfrak{V}}(t, i)$ for every agent by

$$s_i(t, \mathfrak{V}) := \begin{cases} s(E_i^1), & \text{if } t \leq \frac{1}{n} \\ 1, & \text{else.} \end{cases}$$

Proportional serial reservation (PSR) is the algorithm obtained by using these $s_i(t, \mathfrak{V})$ as climbing speed functions of serial reservation.

Being a serial reservation algorithm, PSR is DL-efficient.

Theorem 3.2 (Efficiency)
PSR *is* DL-*efficient.*

Proof. Follows directly from [AS14, Theorem 8]. □

Before establishing strategyproofness of PSR, we will prove its proportionality.

Lemma 3.3
For any input vote $\mathfrak{V}$, the first freeze of PSR *happens at $t \geq \frac{1}{n}$.*

Proof. Let $\mathfrak{V}$ be arbitrary. By Theorem 2.13, there always exists a weakly proportional p. This p is an eligible lottery up until $t = \frac{1}{n}$: Let E be one of the first freezing towers and let t be the time of this occurring. Assume $t < \frac{1}{n}$, then $\text{ceil}_t(E) = t \cdot \text{speed}_0(E) < \frac{1}{n} s(E) = p(E)$, which contradicts the freezing of E. As E was chosen arbitrary, no tower freezes at time $t < \frac{1}{n}$. □

It should be noted that for any n there exist preference profiles for which the first freeze happens as late as $t = 1$. Let n arbitrary, chose $A = \{a, a_1, \ldots, a_n\}$ and define the profile $\mathcal{R}$ as follows:

$$\begin{aligned}
1 &: \{a, a_1\}, \{a_2, \ldots, a_n\} \\
2 &: \{a, a_2\}, \{a_1, a_3, \ldots, a_n\} \\
&\vdots \\
n &: \{a, a_n\}, \{a_1, \ldots, a_{n-1}\}
\end{aligned}$$

As for every $i \in N$ one has $s(E_i^1) = 1$, PSR is equivalent to ESR for this profile. The resulting lottery $p = 1 \cdot a$ fulfils all constraints until $t = 1$, thus no freezing happens beforehand.

As an immediate consequence of this lemma, weak proportionality can be proven.

Theorem 3.4 (Proportionality)
PSR *fulfils weak but not strong proportionality.*

Proof. By Lemma 3.3, no tower freezes prior to $t = \frac{1}{n}$. At that time, the generated constraints already imply weak proportionality of p: For every agent i one has $\text{ceil}_{\frac{1}{n}}(E_i^1) = \frac{s_i(0)}{n} = \frac{s(E_i^1)}{n}$ which means that for every $p \in \text{PSR}(\mathcal{R})$ it is ensured that $p(E_i^1) \geq \frac{s(E_i^1)}{n}$.

A counterexample to strong proportionality is given by the following profile:

$$\begin{aligned}
1 &: \{a, b\}, \{c, d\} \\
2 &: \{b, c\}, \{a, d\} \\
3 &: \{d\}, \{a, b, c\}
\end{aligned}$$

As $s(E_i^1) = 1$ for all $i \in N$, $\text{ESR}(\mathcal{R}) = \text{PSR}(\mathcal{R}) = \{\frac{1}{2}b + \frac{1}{2}d\}$, but strong proportionality requires $p(\{a, b, c\}) = \frac{2}{3}$. □

Corollary 3.5
PSR *is monotonic on the strict domain.*

Proof. Follows directly from Corollary 2.21 □

Now we establish strategyproofness of PSR. As the algorithm is not too different from ESR, one might assume that the level of strategyproofness is similar to ESR's strategyproofness, which is indeed the case.

Theorem 3.6 (Strategyproofness)
PSR *is*

- *strongly* SD-*SP on the strict domain,*
- *weakly* ST-*SP in general, but not weakly* SD-*SP.*

Proof. Strategyproofness on the strict domain follows directly from Corollary 2.21. ST-strategyproofness follows from Theorem 3.4 (PSR is weakly proportional), Corollary 2.14 (weak proportionality implies fair outcome share) and Theorem 2.7 (fair outcome share implies ST-SP).

Now, a counterexample to weak SD-strategyproofness is constructed. Let the preference profile $\mathcal{R}$ be defined by

$$1 : \{e\}, \{a, b\}, \{c, d\} \qquad 3 : \{e\}, \{a\}, \{b, c, d\}$$
$$2 : \{e\}, \{c\}, \{a, b, d\} \qquad 4 : \{a, b, c, d\}, \{e\}$$

PSR yields $p = \frac{3}{4}e + \frac{1}{8}a + \frac{1}{8}c$. If agent 1 reports $\{e\}, \{b\}, \{a, c, d\}$ instead, then $p' = \frac{3}{4}e + \frac{1}{12}a + \frac{1}{12}b + \frac{1}{12}c$, which is strictly better in the SD-sense. □

One may notice the similarities of the counterexample used in this proof and the one used to disprove weak SD-strategyproofness of ESR (see Theorem 1.38). As PSR differs from ESR only during the "first phase", it inherits many properties of it. The following construction demonstrates the idea employed in the above proof: For some agent $i \in N$ let $\mathcal{R}$, $\mathcal{R}'$ be a pair of profiles violating strategyproofness of ESR. Define a dummy element $e \notin A$ and a dummy agent $x \notin N$. Construct $\tilde{R}$, $\tilde{R}'$ s.t. for all $j \in N$ by defining $\tilde{E}_j^{k+1} = E_j^k$ and $E_j^1 = \{e\}$ (analogous for $\mathcal{R}'$). For the new dummy agent x, choose $\tilde{E}_x^1 = \bigcup_{j \in N} E_j^1$ and $\tilde{E}_x^2 = \{e\}$. The "first phase" of PSR reserves $p(e) = n/(n+1)$ and $p(A) = 1/(n+1)$. After this, the constraints generated during the execution of ESR and scaled by $1/(n+1)$ are produced. This idea potentially can be used to prove or disprove further properties.

Proving the monotonicity of PSR remains an open problem. Nevertheless, PSR is outperforming ESR in terms of fairness while retaining its efficiency and strategyproofness. Especially being equivalent to RD on the strict domain is a striking point. In our opinion PSR thus seems to be preferable to ESR in most cases. But alas, the algorithm is only weakly proportional and still only satisfies a weak notion of strategyproofness. The next algorithm is motivated by these motifs.

3.2 Superset SR

As suggested by [ABB13a; ABB14], strategyproofness and efficiency seem to be incompatible to a certain degree. It thus seems unlikely that sticking to DL-efficiency and therefore to serial reservations can lead to an at least weakly SD-strategyproof correspondence. *Superset SR* (SSR) works similarly as ESR in the way of gradually restricting eligible lotteries.

Algorithm 3: Superset simultaneous reservation

Input : $\mathfrak{V} = (N, A, \mathcal{R})$: A vote

Output : The set of lotteries $\text{SSR}(\mathcal{R})$

Symbols: $c : 2^A \to \mathbb{R}$: Current ceiling height of each tower
$s : 2^A \to \mathbb{R}$: The speed at which each tower is pushed
$k : N \to \mathbb{N}$: Number of the equivalence class each agent is in
$\texttt{supersets} : 2^A \to 2^{2^A}, S \mapsto \{S' \subset A \mid S \subseteq S'\}$
$N' \subset N$: Currently active agents
$F \subset 2^A$: Set of all frozen towers

foreach $S \subset A$ **do** $c(S) \leftarrow 0$
foreach $i \in N$ **do** $k(i) \leftarrow 1$
$N' \leftarrow N$
$F \leftarrow \{\emptyset\}$
while $N' \neq \emptyset$ **do**
  foreach $S \subset A$ **do** $s(S) \leftarrow 0$
  foreach $i \in N'$, $S \in \texttt{supersets}(E(i)) \setminus F$ **do** $s(S) \leftarrow s(S) + 1$
  // computeLambdaWithSpeed computes how long towers can be pushed and which towers freeze given the constant speed function s. The procedure can be found on Page 50.
  $(\lambda^\star, S^\star) \leftarrow \texttt{computeLambdaWithSpeed}(A, c, s)$
  $F \leftarrow F \cup S^\star$
  foreach $S \in 2^A \setminus \emptyset$ **do** $c(S) \leftarrow c(S) + s(S) \cdot \lambda^\star$
  foreach $i \in N'$ **do**
    while $E_i^{k(i)} \in F \setminus \emptyset$ **do**
      $k(i) \leftarrow k(i) + 1$
      if $E_i^{k(i)} = \emptyset$ **then** $N' \leftarrow N' \cup \{i\}$ // Agent i is finished
    end
  end
end
return *All lotteries p with $p(S) \geq l(S)$ for all $S \subset A$*

Instead of only climbing the tower of their current equivalence class agents push the ceiling of all towers which are a superset of the tower they are currently in. Additionally, agents are allowed to cooperate, i.e. when they push a ceiling together their speeds add up, and jump up to the ceiling as soon as they enter the tower. Again, towers are frozen when the existence of a lottery satisfying all lower bounds is threatened. Upon dropping down from their primary tower, agents stop pushing all supersets, too, and move to the next equivalence class of theirs, pushing it and all it's supersets.

Similarly to PSR, a freezing time lemma is proven to show proportionality.

Lemma 3.7
The first freeze of SSR *happens at* $t = \frac{1}{n}$.

Proof. Let $D \subset A$ be one of the first freezing towers and t the time of freeze. The tower A gets pushed by every agent, thus moves with speed n and freezes at time $t = \frac{1}{n}$, therefore $t \leq \frac{1}{n}$. Now assume $t < \frac{1}{n}$. Choose a vector $c \in A^n$ such that $\forall\, i \in N : c_i \in E_i^1$. Define $p = \sum_{i \in N} \frac{1}{n} c_i$. One has that $\text{ceil}_t(D) = t \cdot \text{speed}_0(D) < \frac{1}{n} \cdot \text{speed}_0(D)$. By definition of SSR, there have to be $\text{speed}_0(D)$ many agents which are pushing it, i.e. there exists an $I \subset N$ with $|I| = \text{speed}_0(D)$ and $i \in I \Rightarrow c_i \in E_i^1 \subset D$. But by construction $p(D) \geq \frac{1}{n} \cdot |I| > \text{ceil}_t(D)$, implying that tower D can still be pushed. As D was chosen arbitrary, there can be no tower freezing prior $t = \frac{1}{n}$. □

With this lemma, strong proportionality is easily shown.

Corollary 3.8
SSR *fulfils strong proportionality*

Proof. Let $E \subset O$. By definition of SSR $\text{speed}_0(E) = \left|\{i \in N : E_i^1 \subset E\}\right| = s(E)$. By Lemma 3.7, tower E does not freeze before $t = \frac{1}{n}$, thus $\frac{1}{n} \cdot s(E) = \frac{1}{n} \cdot \text{speed}_0(E) = \text{ceil}_{\frac{1}{n}}(E) \leq p(E)$. □

Strong proportionality can now be used to easily prove both efficiency and strategyproofness.

Corollary 3.9 (Properties on the strict domain)
SSR *is* DL-*efficient, strongly* SD-*strategyproof and monotonic for strict preferences.*

Proof. Follows directly from Corollary 2.21. □

Theorem 3.10 (Efficiency and strategyproofness)
SSR *is* ST-*efficient and weakly* ST-*strategyproof on the full domain, but is neither* SD-*efficient nor weakly* SD-*strategyproof.*

Proof. By virtue of Theorem 2.7, ST-efficiency and weak ST-strategyproofness is a direct consequence of fair outcome share, which SSR fulfils by Corollary 3.8 and Corollary 2.14.

As a counterexample to both SD-efficiency and -strategyproofness, consider the following preference profile:

$$1 : \{a, d\}, \{b\}, \{c\} \qquad 3 : \{b\}, \{a, d\}, \{d\}$$
$$2 : \{b, c\}, \{d\}, \{a\}$$

Given this profile, SSR yields the lottery $p = \frac{2}{15}a + \frac{7}{12}b + \frac{1}{12}c + \frac{1}{5}d$. But one can easily verify that $q = \frac{2}{15}a + \frac{2}{3}b + \frac{1}{5}d$ SD-dominates p, as no agent prefers c over b and some prefer b over c. Additionally, if agent 1 instead reports the preference $\{a\}, \{d\}, \{b\}, \{c\}$, SSR yields $p' = \frac{1}{3}a + \frac{2}{3}b$, which he SD-prefers over p, contradicting weak SD-strategyproofness. □

Another unsatisfactory property of SSR is its high computational cost due to its exponential nature (we conjecture that the underlying problem of computing a SSR-solution is NP-hard). Additionally, it is not known if SSR is monotonic. Altogether, improving on the proportionality seems to have come at quite a significant trade-off. The following algorithm tries to remedy this fact while retaining at least strong proportionality by combining the ideas of PSR and SSR.

3.3 Strong Proportional SR

Again, the basic idea of gradually restricting eligible lotteries is utilized. But now, instead of restricting $\triangle(A)$ only lotteries which already satisfy strong proportionality are considered. The original ESR procedure then is executed on this set. By Theorem 2.13 there always exists a strongly proportional lottery and hence this algorithm never returns an empty set.

Algorithm 4: Strong proportional simultaneous reservation

Input : $\mathfrak{V} = (N, A, \mathcal{R})$: A vote

Output : The set of lotteries SPSR($\mathfrak{V}$)

Symbols : $c : 2^A \to \mathbb{R}$: Current ceiling height of each tower,
$h : N \to \mathbb{R}$: Current height of each agent,
$k : N \to \mathbb{N}$: Number of the equivalence class each agent currently is in,
$E : N \to 2^A, i \mapsto E_i^{k(i)}$: The equivalence class each agent is currently in, backed by k

foreach $E \subset A$ **do** $c(E) \leftarrow \frac{s(E)}{n}$ `// Initialize tower heights`
foreach $i \in N$ **do** $h(i) \leftarrow \frac{s(E_i^1)}{n}$, $k(i) \leftarrow 1$ `// Initialize agent data`
$N' \leftarrow N$ `// Currently active agents`
while $N' \neq \emptyset$ **do**
  $(\lambda^\star, N^\star) \leftarrow$ `computeLambda(`A, N', h, c`)`
  foreach $i \in N$ **do**
    $c(E(i)) \leftarrow \max\{c(E(i)), h(i) + \lambda^\star\}$
    if $i \in N^\star$ **then**
      if $E_i^{k(i)+1} = \emptyset$ **then**
        $N' = N' \setminus \{i\}$
      else
        $h(i) \leftarrow 0$
        $k(i) \leftarrow k(i) + 1$
      end
    else
      $h(i) \leftarrow h(i) + \lambda^\star$
    end
  end
end
return *All lotteries p with $p(E) \geq c(E)$ for all $E \subset A$*

We only show some very immediate properties of SPSR.

Corollary 3.11 (Proportionality)
SPSR *is strongly proportional.*

Proof. Follows directly from the definition. □

Similar to SSR, efficiency and strategyproofness are shown using the strong proportionality. Unfortunately no further positive results compared to SSR could be achieved.

Corollary 3.12 (Properties on the strict domain)
SPSR *is* DL-*efficient, strongly* SD-*strategyproof and monotonic on the strict domain.*

Theorem 3.13 (Efficiency and strategyproofness)
SPSR *is* ST-*efficient and weakly* ST-*strategyproof but not weakly* SD-*strategyproof.*

Proof. ST-efficiency and -strategyproofness again follow from fair outcome share. To disprove SD-strategyproofness the same idea which was used for PSR can be used again (actually SPSR gives the exact same results for the profiles used in the proof of Theorem 3.6). □

One might hope that the missing strategyproofness may allow for a better efficiency, which remains to be shown. Empirical data shows that SPSR may indeed be very efficient, usual counterexamples to SD- and DL-efficiency are handled well by the algorithm.

Conjecture 3.14 (Efficiency)
SPSR *is* DL-*efficient.*

As SPSR is not immediately characterizable through speed functions, DL-efficiency has to be proven differently. The consequences of proving this statement right or wrong may be very interesting: If SPSR should indeed be DL-efficient then by virtue of the serial reservation theorem there exist speed functions which capture the essence of strong proportionality in conditions on the primary preference classes alone. This may be used to further understand proportionality in general. Proving it wrong may reveal a general incompatibility between DL-efficiency and strong proportionality, giving an incentive to refine the idea behind proportionality. If at least SD-efficiency is proven for this algorithm, SPSR seems to be a highly attractive algorithm in cases where strategyproofness is not of utmost importance.

Chapter 4

Conclusion

	Efficiency			Strategyproofness				Proportional		
	ST	SD	DL	w-ST	w-SD	DL	s-SD	weak	strong	Mon.
RSD	+	$-^s$	$-^s$	+	+	+	+	+	+	+
MR	+	$-^s$	$-^s$	+	+	+	$-^s$	+	+	+
SML	+	+	−	+	−	−	−	−	−	−
ESR	+	+	+	+	$-^s$	$-^s$	−	−	−	+
PSR	+	+	+	+	$-^s$	$-^s$	$-^s$	+	$-^s$	$?^s$
SSR	+	$-^s$	$-^s$	+	$-^s$	$-^s$	$-^s$	+	+	$?^s$
SPSR	+	$?^s$	$?^s$	+	$-^s$	$-^s$	$-^s$	+	+	$?^s$

+ Fulfils the property
− Does not fulfil the property in general
? Not known
s Fulfils the property for strict preferences

Table 4.1: Comparison of the presented algorithms

We presented a new way to measure the a-priori fairness an algorithm offers to agents in the form of proportionality, contributing to a sparsely researched field. Proportionality is compatible with random dictator on the strict domain and even characterises it completely. It is fulfilled by some major SDS on the general domain but not all. Highlighting some properties and flaws of the new definition, we additionally presented some ideas on how to further extend the underlying concept of proportionality. In an effort to find a SDS which is proportional and DL-efficient we defined proportional serial reservation, based on ESR introduced by Aziz and Stursberg [AS14]. PSR indeed is DL-efficient but unfortunately only weakly proportional. It seems to be favourable over ESR in most cases, fulfilling the same levels of efficiency and strategyproofness but additionally satisfying proportionality. Trying to find strongly proportional derivations of serial reservation yielded only mediocre results. both SPSR and SSR have a fair amount of weaknesses. A number of questions remain open: Adding to the yet unknown efficiency of SPSR, monotonicity is not yet proven nor disproven for the new algorithms. Also we have argued that, while certainly being interesting properties, neither strong nor weak proportionality seem to fully capture fine grained group dynamics in voting. Answering any of these questions may further insight into fairness, proportionality and the underlying difficulties.

Appendix A

Supplements

A.1 Proof of ESR monotonicity

Proof. Let the vote $\mathfrak{V}$, agent $i \in N$ and alternative $a \in A$ be arbitrary. Let $k \in \mathbb{N}$ and $E \subset A$ such that $E_i^k = E \cup \{a\}$ (Note that E might be empty or $k = 1$). The two most elementary methods agent i can employ to reinforce a compared to $\mathcal{R}$ is to either

1. "move" it in between E_i^k and E_i^{k-1} or
2. put it into E_i^{k-1},

as illustrated below. Every other kind of reinforced profile can be constructed by iteratively applying these two steps.

$$\dots, E_i^{k-1}, E \cup \{a\}, \dots \begin{array}{l} \xrightarrow{1.} \dots, E_i^{k-1}, \{a\}, E, \dots \\ \xrightarrow[2.]{} \dots, E_i^{k-1} \cup \{a\}, E, \dots \end{array}$$

Dealing with Item 1 first, employing a proof by contradiction. Let $\mathcal{R}'$ be defined as illustrated. W.l.o.g. one can assume that E is not empty, otherwise $\mathcal{R} = \mathcal{R}'$. Suppose monotonicity is violated for this preference $\mathcal{R}'$, i.e. there exists a $p \in \text{ESR}(\mathfrak{V})$ with $p(a) > q(a)$ for all $q \in \text{ESR}(\mathfrak{V}')$. Let now t be the time of agent i dropping off E_i^{k-1}, which is the same for both inputs. If $k = 1$, then $t = 0$. Analogously, let t'_a be the time of i moving on from $\{a\}$ for profile $\mathcal{R}'$ and similarly t_E the corresponding time for tower E_i^k and profile $\mathcal{R}$. Note that $t'_a \leq t_E$, i.e. agent i will not drop off E_i^k earlier as t'_a, any lottery satisfying $p(a) \geq \tau$ also satisfies $p(E_i^k) = p(a) + p(E) \geq \tau$. At time t all tower heights are the same for both algorithm executions, simply because the process did not "see" any difference up until then. $\{a\}$ might have been already frozen at time t at some height $h > 0$ (e.g. by another agent, which liked a the most), but then, by the same argumentation, $p(a) = q(a) = h$. Thus one can assume w.l.o.g. that $\{a\}$ freezes at t'_a.

The central idea now is to show that between times t and t'_a no other agent behaves differently in the sense that no agent drops from some tower at different times. As a consequence, all constraints except the ones of $\{a\}$ and E_i^k stay the same for both inputs, yielding the contradiction. Suppose there is an agent j, which dropped from tower E_j at different times t_j and t'_j for the two inputs. Let $\tilde{t}_j$ be the smaller of the two. By the previous argumentation all agents behave equally until time t, thus $t < \tilde{t}_j$. Let further $\tilde{t}_j < t'_a$, otherwise the agent would not be of further interest. Assume that j, t_j and t'_j are chosen in a way that $\tilde{t}_j$ is the first time of this happening. If E_j was frozen beforehand already and j only climbed to it's top to bounce off of it, he would not behave differently, as E'_j's ceiling would have been fixed at the same height for both inputs (by assumption towers are frozen at equal heights until time $\tilde{t}_j$). Thus E_j freezes at time t_j for input $\mathfrak{V}$ and t'_j for $\mathfrak{V}'$. The cases $t_j < t'_j$ and $t'_j < t_j$ have to be handled separately:

- $t_j < t'_j$, implying that E_j froze earlier for input $\mathfrak{V}$. As argued, all tower heights except the ones i modified are equal, hence any q is an eligible profile for input R at time t_j. But, as E_j did not freeze for input $\mathfrak{V}'$ and q is a result of invoking ESR on this input, $q(E_j) > \text{ceil}_{t_j}(E_j)$, contradicting the freeze at t_j.
- $t'_j < t_j$: Again, at $\tilde{t}$ towers are of equal height except E_i^k and $\{a\}$. Note that $\{a\}$ is not frozen for input $\mathfrak{V}'$, as $t'_j = \tilde{t}_j < t'_a$ was assumed. Thus there exists a $p' \in \text{ESR}(\mathfrak{V})$, which satisfies $p(a) \geq \text{ceil}_{t'_j}(\{a\})$ and the other currently active constraints, otherwise $\{a\}$ would necessarily freeze. As $t_j > t'_j$, E_j was pushed further, i.e. $p(E_j) > \text{ceil}'_{t'_j}(E_j)$, again contradicting the freeze of E_j.

Therefore such an agent j does not exist and at time t'_a all towers are at the same height except for $\{a\}$ and E_i^k. t'_a was chosen as the time when agent a drops off $\{a\}$ and it was argued previously why one can assume that $\{a\}$ freezes exactly at that time. $\{a\}$ being frozen now implies that there exists no $q \in \triangle(A)$ which satisfies all current constraints and has $q(a) > \text{ceil}'_{t'_a}(\{a\})$. However $p(a) > q(a)$ for all $q \in \text{ESR}(\mathfrak{V}')$ by assumption. As just argued, at time t'_a all constraints except E_i^k and $\{a\}$ are equal. With p being an outcome of $\text{ESR}(\mathfrak{V})$, p indeed satisfies all constraints at time t'_j for input $\mathfrak{V}'$. This is a contradiction to $\{a\}$ freezing at t'_a.

Item 2 is handled analogously. Let again $\mathcal{R}'$ be as illustrated, i.e. agent i moved a up into E_i^{k-1} and therefore $E'^{k-1}_i = E_i^{k-1} \cup \{a\}$. Let now t (t') be the time when agent i drops from E_i^{k-1} (E'^{k-1}_i) for input $\mathfrak{V}$ ($\mathfrak{V}'$). Certainly $t \leq t'$, as $E_i^{k-1} \subset E'^{k-1}_i$ and therefore its "easier" for lotteries to satisfy the corresponding constraint. If $t = t'$, i.e. both of the towers are frozen at the same height, then there exists no lottery q such that q fulfils all constraints and has $q(a) > 0$, otherwise $E'^{k-1}_i = E_i^{k-1} \cup \{a\}$ could grow further. As the same constraints are active for input $\mathfrak{V}$, there also can not exist a $p \in \text{ESR}(\mathfrak{V})$ with $p(a) > 0$. So, by climbing E'^{k-1}_i further past time t, agent i reserves more and more probability for a (all in all $t' - t$). Let t_k be the time of i dropping of E_i^k for input $\mathfrak{V}$. Assume once more that there exists an agent j and tower E_j which drops off at different times t_j and t'_j with $t < \tilde{t}_j < t'$, w.l.o.g. $\tilde{t}_j$ is assumed to be the first time of this occurring. Again, two cases have to be distinguished:

- $t_j < t'_j$: For input $\mathfrak{V}$ agent i currently is pushing E_i^k, for $\mathfrak{V}'$ he is pushing $E_i^{k-1} \cup \{a\}$, none of which are freezing by assumption. As previously argued, i pushing $E_i^{k-1} \cup \{a\}$ is equivalent reserving probability for $\{a\}$ since t. A resulting lottery q therefore fulfils $q(a) \geq t_j - t$. As E_i^k currently is at the same height $t_j - t$, q is eligible for $\mathfrak{V}$ at time t_j, but $q(E_j) > \mathrm{ceil}_{t_j}(E_j)$.

- $t'_j < t_j$: By similar argumentation as above, there exists a $p \in \mathrm{ESR}(\mathfrak{V})$, which fulfils $p(a) \geq t_j - t$, i.e. p is eligible for $\mathfrak{V}'$ at time t'_j, too. As E'_j is not frozen at t'_j for input $\mathfrak{V}$ one has $p(E'_j) > \mathrm{ceil}'_{t'_j}(E'_j)$.

Together, no such agent j exists. This again means that at time t' all constraints except those of the two modified sets are equal, meaning that p is an eligible lottery at time t' for input $\mathfrak{V}'$. But by assumption $p(a) > q(a)$, thus the tower ${E'}_i^{k-1}$ could grow further. □

A.2 Auxiliary Procedures

Function computeLambdaWithSpeeds(A, c, s), subroutine of SSR

Input : A: The set of alternatives,
$c : 2^A \to \mathbb{R}$: Current ceiling heights,
$s : 2^A \to \mathbb{R}$: Pushing speeds

Output : $\lambda^\star$: Time pushing can happen until some tower freezes,
$S^\star$: Towers that freeze after pushing for $\lambda^\star$

Solve the following LP:

$$\begin{aligned} \lambda^\star &\leftarrow \max \lambda \\ \sum_{a \in S} p_a &\geq c(S) + s(S) \cdot \lambda \quad \forall\, S \subset A \\ \sum_{a \in A} p_a &\leq 1 \\ p_a &\geq 0 \quad \forall\, a \in A \\ \lambda &\geq 0 \end{aligned}$$

$S^\star \leftarrow \{\}$ `// The set of towers which will freeze`

foreach $S \subset A : s(S) > 0$ **do**

  Solve the following LP:

$$\begin{aligned} \tilde{\lambda} &\leftarrow \max \sum_{a \in S} p_a - (c(S) + s(S) \cdot \lambda^\star) \\ \sum_{a \in S} p_a &\geq c(S) + s(S) \cdot \lambda^\star \quad \forall\, S \subset A \\ \sum_{a \in A} p_a &\leq 1 \\ p_a &\geq 0 \quad \forall\, a \in A \end{aligned}$$

  if $\tilde{\lambda} = 0$ **then** $S^\star \leftarrow S^\star \cup \{S\}$

end

return $\lambda^\star, S^\star$

Function computeLambda(A, N, h, c), subroutine of ESR

Input : N: The set of agents,
A: The set of alternatives,
$h : N \to \mathbb{R}$: Current agent heights,
$c : 2^A \to \mathbb{R}$: Current tower heights

Output : $\lambda^\star$: The time agents can push until some of them drop off their tower,
$N^\star$: The set of agents which drop off their towers after pushing for $\lambda^\star$

Solve the following LP:

$$\lambda^\star \leftarrow \max \lambda$$
$$\sum_{a \in S} p_a \geq c(S) \quad \forall S \subset A \quad \texttt{// Satisfy all active tower heights}$$
$$\sum_{a \in E_i^{k(i)}} p_a \geq h(i) + \lambda \quad \forall i \in N \quad \texttt{// Satisfy climbing constraints}$$
$$\sum_{a \in A} p_a \leq 1 \quad \texttt{// Ensure } p \texttt{ is a lottery}$$
$$p_a \geq 0 \quad \forall a \in A$$
$$\lambda \geq 0$$

`// Now determine which agents actually bounce after pushing` $\lambda^\star$

$N^\star \leftarrow \{\}$ `// The set of agents which will bounce off`

foreach $i \in N$ **do**

  Solve the following LP:

$$\tilde{\lambda} \leftarrow \max \sum_{a \in E_i^{k(i)}} p_a - (h(i) + \lambda^\star)$$
$$\sum_{a \in S} p_a \geq c(S) \quad \forall S \subset A$$
$$\sum_{a \in E_i^{k(i)}} p_a \geq h(i) + \lambda^\star \quad \forall i \in N$$
$$\sum_{a \in A} p_a \leq 1$$
$$p_a \geq 0 quad \forall a \in A$$

  if $\tilde{\lambda} = 0$ **then** $N^\star \leftarrow N^\star \cup \{i\}$

end

return $\lambda^\star, N^\star$

Appendix B

Implementation

The algorithms ESR, PSR, SPSR and SSR have been implemented using Python. The complete code can be found on Github under https://github.com/incaseoftrouble/social_choice_fairness. In this section the code structure will be explained briefly.

The module `society` contains numerous classes used to represent voting problems. The class `Choice` wraps any given object and gets bundled into a `ChoiceClass`, which represents an equivalence class. A list of `ChoiceClasses` is used to create a `Preference` object, required to create an `Agent`. At last, multiple `Agent` objects lead to a `Vote`, containing all information of a social choice problem. `Lottery` is a simple object to represent a full probability distribution over any kind of objects. Additionally, as assignment instances can be transformed to social choice problems, the classes `Assignment` and `AssignmentLottery` have been created. All classes in this module validate their inputs and ensure that no data inconsistency occurs. To easily create `Vote` objects, `parser.py` provides the function `parseVoteFromDict` and `toAssignmentVote`. An exemplary usage is as follows:

```
from vote.parser import parseVoteFromDict

if __name__ == '__main__':
  vote = parseVoteFromDict({
    1: ["a", ("b", "c")],
    2: ["b", "a", "c"],
    3: [("a", "c"), ("b")],
  })
  print str(vote)
```

This code snippet produces the following output:

```
Agents: 1,2,3; Choices: a,b,c
  1: (a),(b,c)
  2: (b),(a),(c)
  3: (a,c),(b)
```

The package `solvers` contains all implemented solvers. They all rely on the PuLP package (https://pythonhosted.org/PuLP/) to solve the occurring LP optimization problems. As PuLP provides an excellent abstraction layer to the actual solving process, many different LP solvers can be used. To deal with floating point inaccuracies, the `SolverSettings` object was created. It provides methods like `isClose(a, b)`, which checks if `a` and `b` are equal up to some tolerances, or `isInInterval(value, a, b)`, testing if `value` is inside the interval [`a`, `b`] up to some tolerance. The actual solvers are contained in the modules `solver.sr` and `solver.ssr`. Additionally to the solver methods, the classes `Tower`, `AgentData` and `SRState` can be found in `solver.sr`. `Tower` and `AgentData` provide low-level functions to deal with climbing agents and pushed tower ceilings, `SRState` stores all relevant informations generated during the solving process of a SR-like algorithm and provides methods to perform actions like agent-climbing in a consistent manner. Given a state, `computeLambda` computes the maximal climbing time until some agent bounces and also determines which agents bounce. Should the speed functions be non-constant but only piecewise constant, the optional parameter `maximumTime` can be used to limit lambda until the next change in speed. Dealing with speed functions that are not piecewise constant is not possible with this framework, as this requires solving a non-linear problem. The three implemented SR-like solvers are - aside from the initialization - working completely similar:

```
state = SRState(vote, solverSettings)

# Do further initialization here
# ...

while not state.isFinished():
  (climbTime, bouncingAgents) = computeLambda(state)
  state.advance(climbTime, bouncingAgents)
return findLottery(vote, state.getCurrentClassHeights(), ↷
    ↪ state.getSettings())
```

`solver.ssr` provides an implementation of SSR. Similarly structured, this module contains the classes `Tower`, `AgentData` and `SSRState` together with the function `computeLambda`. Note that here the only state agents have is the equivalence class they currently are pushing and height is an exclusive property of towers.

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